FRACTIONAL PATH INTEGRAL IN MOMENTUM SPACE IN TIME EVOLUTION TECHNIQUE

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Abstract

A fractional path integral (FPI) in momentum space are defined in term of White noise, we used time evolution technique to transform FPI into momentum space. The free particle and simple harmonic oscillator path integrals have been developed by our FPI in momentum space, then we compare it with classical path integral.

Keyword : Fractional path integral, momentum space, Kernal (propagator), time evolution, Fractional Hamilton.

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Introduction

The importance of fractional calculus is that it implies the results more general and more precise which make the classical models are special case of fractional models. The fractional calculus are very helpful for mathematicians, engineers and scientists (Kumar & Baleanu, 2019), fractional calculus have many application in physical system as continuous third order system they used fractional derivatives to constructed the Hamiltonian formulation (Alawaideh, Hijjawi, Khalifeh, 2021), using fractional derivative to reform Maxwells equations and electromagnetic Lagrangian density on fractional form.(Jaradat, Hijjawi, Khalifeh, 2012).

The importance of the path integral (PI) formulation of quantum mechanics is coming from the question: If the particle is at a position x at time t = 0, what is the probability amplitude that it will be at some other position x_o at a later time t = T? So, we use the fractional path integral to find the general probability amplitude. Path integrals method does not give a new result for a single particle in quantum mechanics, but we can understand clearly the classical limit of quantum mechanics. The importance of path integrals appears in quantum field theory. Furthermore, the nonperturbative phenomena are most easily showed by path integrals. [(MacKenzie & René-J-A-Lévesque, 1999)].

The advantages of momentum representation in quantum mechanics can be used to emphasize the basic symmetry between the representations, obtain the different nature of operators in the representation. Moreover, finding wave function in momentum space of some physical systems is easier than phase space. [(Núñez-Yépez et al., 2000)].

Path integral is a quantum mechanical formulation with many applications in the fields of: quantum mechanics, quantum field theory, statistical mechanical, and condensed matter physics. [(MacKenzie & René-J-A-Lévesque, 1999)].

A phase space path integral uses Hamiltonians of the general form

$$H(p, x) = T(p) + V(x),$$

where T(p) is kinetic energy $\frac{p^2}{2m}$, explicit quadratic form, and V(x) is the potential energy.[(Blau, 2019)].

In this study we will use the fractional Levy path integral definition in phase space then rewrite it in term of White noise which explicitly depend on momentum p. Then use time evolution method to construct FPI in momentum space. We apply it in some physical systems such as free particle and simple harmonic oscillator.

1-Fractional path integral in momentum space in term of whit noise

The fractional path integral is defined as

$$K_L(x_b t_b|x_a t_a) = \int_{x(t_a)=x_a}^{x(t_b)=x_b} Dx(\tau)\,.\,e^{-\frac{i}{\hbar}\int_{t_a}^{t_b} d\tau V(x(\tau))} \quad (1)$$

Dx(τ) is the fractional path integral given by

$$\int_{x(t_a)=x_a}^{x(t_b)=x_b} Dx(t)\ldots = \lim_{N\to\infty}\int_{x(t_a)=x_a}^{x(t_b)=x_b} dx_1 dx_2 \ldots\ldots dx_{N-1}\hbar^{-N}\left(\frac{iD_\alpha\varepsilon}{\hbar}\right)^{-N/\alpha}\cdot\prod_{j=1}^{N} L_\alpha\left\{\frac{1}{\hbar}\left(\frac{\hbar}{iD_\alpha\varepsilon}\right)^{1/\alpha}[x_j - x_{j-1}]\right\} \quad (2)$$

Where L_α is Levy distribution α is the Levy index $1 < \alpha \le 2$, and D_α is fractional quantum diffusion coefficient have a physical dimension $[D_\alpha] = erg^{1-\alpha}cm^\alpha sec^{-\alpha}$, and $\varepsilon = \frac{t_b - t_a}{N}$, so we can write Eq. 1.1 as

$$K_L(x_b t_b|x_a t_a) = \lim_{N\to\infty}\int_{x(t_a)=x_a}^{x(t_b)=x_b} dx_1 dx_2 \ldots\ldots dx_{N-1}\hbar^{-N}\left(\frac{iD_\alpha\varepsilon}{\hbar}\right)^{-N/\alpha}\cdot\prod_{j=1}^{N} L_\alpha\left\{\frac{1}{\hbar}\left(\frac{\hbar}{iD_\alpha\varepsilon}\right)^{1/\alpha}[x_j - x_{j-1}]\right\}.e^{-\frac{i}{\hbar}\int_{t_a}^{t_b} d\tau V(x(\tau))} \quad (3)$$

(Laskin, 2000)

The momentum trajectory as a Brownian fluctuation $B(\tau)$ can be defined as

$$p(\tau) = p_\circ + \frac{\sqrt{\hbar m}}{t - t_\circ}B(\tau) \quad , \quad 0 \le \tau \le t \quad (4)$$

And the space variable in term of White noise $w(\tau)$,

$$x(\tau) = \sqrt{\frac{\hbar}{m}}(t - t_\circ)w(\tau) \quad , \quad 0 \le \tau \le t \quad (5)$$

Rewrite Eq. 4 and Eq. 5 in term of $B(\tau)$ and $w(\tau)$ respectively

$$B(\tau) = \frac{(t - t_\circ)}{\sqrt{\hbar m}}(p(\tau) - p_\circ), \quad (6)$$

$$w(\tau) = \sqrt{\frac{m}{\hbar}}\frac{x(\tau)}{(t - t_\circ)} \quad (7)$$

(Bock, 2014)

Now derive Eq. 6 and 7 we get

$$\dot{B}(\tau) = \frac{(t - t_\circ)}{\sqrt{\hbar m}}\dot{p}(\tau) \quad (8)$$

$$dw(\tau) = \sqrt{\frac{m}{\hbar}}\frac{dx(\tau)}{(t - t_\circ)} \quad (9)$$

But $w(\tau) = \frac{dB(\tau)}{d\tau} = \dot{B}(\tau)$ so we can write that,

$$\dot{B}(\tau) = w(\tau) = \frac{(t - t_\circ)}{\sqrt{\hbar m}}\dot{p}(\tau) \quad (10)$$

Equation 10 represent white noise in term of momentum now we can rewrite Eq. 3 in momentum space in term of white noise,

$$K_L(x_b t_b|x_a t_a) = \lim_{N\to\infty}\int_{x(t_a)=x_a}^{x(t_b)=x_b}\left(\sqrt{\frac{\hbar}{m}}(t_b - t_a)\right)^{N-1} dw_1 dw_2 \ldots\ldots dw_{N-1}\hbar^{-N}\left(\frac{iD_\alpha\varepsilon}{\hbar}\right)^{-N/\alpha}\cdot\prod_{j=1}^{N} L_\alpha\left\{\frac{1}{\hbar}\left(\frac{\hbar}{iD_\alpha\varepsilon}\right)^{1/\alpha}[w_j - w_{j-1}](\sqrt{\frac{\hbar}{m}}(t_b - t_a))\right\}.e^{-\frac{i}{\hbar}\int_{t_a}^{t_b} d\tau V(\sqrt{\frac{\hbar}{m}}(t-t_\circ)w(\tau))} \quad (11)$$

2-Fractional path integral in momentum space by time evolution

The fractional propagator (Kernal) in momentum space can be written as

$$K_\alpha(p_b, t_b; p_a, t_a) = \langle p_b|U_\alpha(t_b, t_a)|p_a\rangle \quad , \quad t_b > t_a \quad (12)$$

Where $\hat{U}_\alpha(t_b, t_a)$ is the fractional time evolution operator.

Feynman realized that due to the fundamental composition law of the time evolution operator, the fractional propagator could be sliced into a large number, say N + 1, of time evolution operators, each acting across an infinitesimal time slice of thickness $\epsilon = t_n - t_{n-1} = \frac{t_b - t_a}{N+1}$.

$$K_\alpha(p_b, t_b; p_a, t_a) = \left\langle p_b\left|\hat{U}_\alpha(t_b, t_N)\,\hat{U}_\alpha(t_N, t_{N-1})\ldots\hat{U}_\alpha(t_n, t_{n-1})\ldots\hat{U}_\alpha(t_{b2}, t_1)\hat{U}_\alpha(t_1, t_a)\right|p_a\right\rangle \quad (13)$$

Use the complete set (closure relation)

$$\int_{-\infty}^{\infty}\frac{dp}{2\pi\hbar}|p\rangle\langle p| = 1 \quad (14)$$

Put Eq. 14 in Eq. 13 we get

$$K_\alpha(p_b, t_b; p_a, t_a) = \int_{-\infty}^{\infty}\frac{dp}{2\pi\hbar}\int_{-\infty}^{\infty}\frac{dp}{2\pi\hbar}\ldots\left\langle p_b\left|\hat{U}_\alpha(t_b, t_N)|p\rangle\langle p|\,\hat{U}_\alpha(t_N, t_{N-1})\ldots\hat{U}_\alpha(t_n, t_{n-1})|p\rangle\langle p|\ldots\hat{U}_\alpha(t_{b2}, t_1)|p\rangle\langle p|\hat{U}_\alpha(t_1, t_a)\right|p_a\right\rangle \quad (15)$$

The fractional propagator becomes a product of N-integrals

$$K_\alpha(p_b,t_b;p_a,t_a) = \prod_{n=1}^{N}\left[\int_{-\infty}^{\infty} dp_n\right]\prod_{n=1}^{N+1}(p_nt_n|p_{n-1}t_{n-1})$$

the fractional time evolution operator,

$$\hat{U}_\alpha(t_b,t_{a)} = e^{-i\epsilon H_{\alpha,\beta}(x,p,t)/\hbar} = e^{-i\epsilon(T_\alpha+V_\beta)/\hbar} \quad (16)$$

Where

$H_{\alpha,\beta} = \hat{T}_\alpha + \hat{V}_\beta$ is fractional Hamiltonian and $\hat{T}_\alpha$ is fractional kinetic energy and $\hat{V}_\beta$ is fractional potential energy ,

$$\hat{T}_\alpha = D_\alpha|p|^\alpha = D_\alpha|\hbar\nabla|^\alpha \quad ,\hat{V}_\beta = q^2|x|^\beta = q^2\left|-i\hbar\frac{d}{dp}\right|^\beta$$

The quantum Riesz fractional derivative $(\hbar\nabla)^\alpha$ can be define as Fourier transform

$$(\hbar\nabla)^{\alpha} = -\frac{1}{2\pi\hbar}\int_{-\infty}^{\infty} dp e^{ipx/\hbar}|p|^\alpha\varphi(p,t)$$

the q^2 have a physical dimension $[q^2] = erg^{1/2}cm^{-\beta/2}$ (Laskin,2002)

$$(p_nt_n|p_{n-1}t_{n-1}) = \langle p_n|e^{-i\epsilon H_{\alpha,\beta}(p,x,t)}|p_{n-1}\rangle$$

$$\langle p_n|e^{-i\epsilon H_{\alpha,\beta}(p,x,t)}|p_{n-1}\rangle \approx \int_{-\infty}^{\infty}\frac{dp}{2\pi\hbar}\left\langle p_n\left|e^{-i\epsilon V_\beta(x,t_n)/\hbar}\right|p\right\rangle\left\langle p\left|e^{-i\epsilon T_\alpha(p,t_n)/\hbar}\right|p_{n-1}\right\rangle \quad (17)$$

The first bracket can be found as

$$\left\langle p_n\left|e^{-i\epsilon V_\beta(x,t_n)/\hbar}\right|p\right\rangle = \delta(p_n-p)e^{-i\epsilon V_\beta(x,t_n)/\hbar}$$

And the second

$$\left\langle p\left|e^{-i\epsilon T_\alpha(p,t_n)/\hbar}\right|p_{n-1}\right\rangle = \int_{-\infty}^{\infty} dx_n e^{-ix_n(p_{n+1}-p_n)/\hbar}e^{-i\epsilon T_\alpha(p,t_n)/\hbar}$$

Put the previous two equation into Eq. 17 we get,

$$K_\alpha(p_b,t_b;p_a,t_a) = \prod_{n=1}^{N}\left[\int_{-\infty}^{\infty}\frac{dp_n}{2\pi\hbar}\right]\prod_{n=0}^{N}\left[\int_{-\infty}^{\infty}dx_n\right]e^{-ix_n(p_{n+1}-p_n)/\hbar}e^{-i\epsilon T_\alpha(p,t_n)/\hbar}e^{-i\epsilon V_\beta(x,t_n)/\hbar}$$

$$K_\alpha(p_b,t_b;p_a,t_a) = \prod_{n=1}^{N}\left[\int_{-\infty}^{\infty}\frac{dp_n}{2\pi\hbar}\right]\prod_{n=0}^{N}\left[\int_{-\infty}^{\infty}dx_n\right]e^{-ix_n(p_{n+1}-p_n)/\hbar}e^{-i\epsilon(T_\alpha(p,t_n)+V_\beta(x,t_n))/\hbar} \quad (18)$$

$$K_\alpha(p_b,t_b;p_a,t_a) = \prod_{n=1}^{N}\left[\int_{-\infty}^{\infty}\frac{dp_n}{2\pi\hbar}\right]\prod_{n=0}^{N}\left[\int_{-\infty}^{\infty}dx_n\right]e^{-ix_n(p_{n+1}-p_n)/\hbar}e^{-i\epsilon H_{\alpha,\beta}/\hbar} \quad (19)$$

$$K_\alpha(p_b,t_b;p_a,t_a) = \prod_{n=1}^{N}\left[\int_{-\infty}^{\infty}\frac{dp_n}{2\pi\hbar}\right]\prod_{n=0}^{N}\left[\int_{-\infty}^{\infty}dx_n\right]e^{\frac{i}{\hbar}(\sum_{n=0}^{N}-x_n(p_{n+1}-p_n)-\epsilon H_{\alpha,\beta}(x,p,t))} \quad (20)$$

The fractional path integral in momentum space in continuum limits $N\to\infty, \epsilon\to 0$

$$K_\alpha(p_b,t_b;p_a,t_a) = \int_{p_a}^{p_b}\frac{Dp}{2\pi\hbar}\int Dx\, e^{\frac{i}{\hbar}\int_{t_a}^{t_b}dt(-x_n\dot{p}-H_{\alpha,\beta}(x,p,t))} \quad (21)$$

$$K_\alpha(p_b,t_b;p_a,t_a) = \int_{p_a}^{p_b}\frac{Dp}{2\pi\hbar}\int Dx\, e^{\frac{i}{\hbar}S_{\alpha,\beta}(p,x,t)} \quad (22)$$

Where, $S_{\alpha,\beta}(p,x,t)$ is fractional action function in momentum space

$$S_{\alpha,\beta}(p,x,t) = \int_{t_a}^{t_b} -\dot{p}(t)x(t) - H_{\alpha,\beta}(p,x,t)\,dt \quad (23)$$

3-Fractional path integral in momentum space for free particles

For free particle the potential energy is zero

$$H_\alpha(p,x,t) = D_\alpha|p|^\alpha$$

The FPI can be find by Eq. 13

$$K_\alpha(p_b,t_b;p_a,t_a) = \langle p_b|\hat{U}_\alpha(t_b,t_N)\,\hat{U}_\alpha(t_N,t_{N-1})\dots\hat{U}_\alpha(t_n,t_{n-1})\dots\hat{U}_\alpha(t_{b2},t_1)\hat{U}_\alpha(t_1,t_a)|p_a\rangle$$

But for free particles $\hat{U}_\alpha(t_b,t_{a)} = e^{-i\epsilon H_\alpha(p)/\hbar} = e^{-i\epsilon(T_\alpha)/\hbar}$

$$K_\alpha(p_b,t_b;p_a,t_a) = \langle p_b|\hat{U}_\alpha(t_b,t_a)\,|p_a\rangle = \langle p_b|e^{-i\epsilon(T_\alpha)/\hbar}\quad|p_a\rangle = \langle p_b|e^{-i\epsilon D_\alpha|p|^\alpha/\hbar}\quad|p_a\rangle$$

Use closure relation,

$$\int_{-\infty}^{\infty} dx_n|x_n\rangle\langle x_n| = 1$$

$$K_\alpha(p_b,t_b;p_a,t_a) = \int_{-\infty}^{\infty} dx_n\,\langle p_b|e^{-i\epsilon D_\alpha|p|^\alpha/\hbar}\quad|x_n\rangle\langle x_n|p_a\rangle$$

$$K_\alpha(p_b,t_b;p_a,t_a) = e^{-i\epsilon D_\alpha|p|^\alpha/\hbar}\int_{-\infty}^{\infty} dx_n\langle p_b|x_n\rangle\,\langle x_n|p_a\rangle$$

$$K_\alpha(p_b,t_b;p_a,t_a) = e^{-i\epsilon D_\alpha|p|^\alpha/\hbar}\quad\langle p_b|p_a\rangle\int_{-\infty}^{\infty} dx_n|x_n\rangle\langle x_n|$$

$$K_\alpha(p_b,t_b;p_a,t_a) = e^{-i\epsilon D_\alpha|p|^\alpha/\hbar}\quad\langle p_b|p_a\rangle$$

Where,

$$\langle p_b|p_a\rangle = 2\pi\hbar\delta(p_b-p_a)$$

The fractional path integral of free particle

$$K_\alpha(p_b,t_b;p_a,t_a) = 2\pi\hbar\delta(p_b-p_a)e^{-i\epsilon D_\alpha|p|^\alpha/\hbar}$$

For continuum limit $N\to\infty$,

$$K_\alpha(p_b,t_b;p_a,t_a) = 2\pi\hbar\delta(p_b-p_a)e^{-i(t_b-t_a)D_\alpha|p|^\alpha/\hbar}$$

For free particles Hamiltonian was independent of x and not explicit depend on t, the Eq. 20 become trivial solution. So, the N + 1 integrals over $x_n(n=0,1,\dots N)$ can be done yielding a product of δ-functions $\delta(p_b-p_N)\dots..\delta(p_1-p_0)$. The integrals over the N momenta $p_n(n=1,\dots.,N)$ are all focused to the initial momentum $p_N = p_{N-1} = \cdots p_1 = p_a$. We take A single a final δ-function $2\pi\hbar\delta(p_b-p_a)$ remains, with $N+1$ factors $\prod_{n=0}^{N} e^{-i\epsilon D_\alpha|p_a|^\alpha/\hbar}$, from Eq. 21, $\dot{p} = \frac{\partial H_\alpha}{\partial x} = 0$

$$K_\alpha(p_b,t_b;p_a,t_a) = \int_{p_a}^{p_b}\frac{Dp}{2\pi\hbar}\int Dx\, e^{\frac{i}{\hbar}\int_{t_a}^{t_b}dt(-\epsilon H_{\alpha,\beta}(x,p,t))}$$

$$K_\alpha(p_b,t_b;p_a,t_a) = e^{-i(t_b-t_a)D_\alpha|p_a|^\alpha/\hbar}\int_{p_a}^{p_b}\frac{Dp}{2\pi\hbar}\int Dx$$

Then the FPI of free particle in momentum space can be written as

$$K_\alpha(p_b,t_b;p_a,t_a) = 2\pi\hbar\delta(p_b - p_a)e^{-i(t_b-t_a)D_\alpha|p|^\alpha/\hbar} \quad \ldots(24)$$

We find from this result that the amplitude we found by the normal method was the same as that found by the fractional path integral method. Take $\alpha = 2$, classical case we return to normal result $D_2 = \frac{1}{2m}$

$$K_\alpha(p_b,t_b;p_a,t_a) = 2\pi\hbar\delta(p_b - p_a)e^{-i(t_b-t_a)\frac{p^2}{2m}/\hbar}$$

4- Fractional path integral in momentum space for simple harmonic oscillator (SHO)

The Hamiltonian of SHO is

$$H(p,x) = \frac{p^2}{2m} + \frac{1}{2}m\omega^2x^2$$

The path integral can be written as,

$$K(x_b,t_b;x_a,t_a) = \prod_{n=1}^{N}\left[\int_{-\infty}^{\infty}\frac{dp_n}{2\pi\hbar}\right]\prod_{n=0}^{N}\left[\int_{-\infty}^{\infty}dx_n\right]e^{\frac{i}{\hbar}S(p,x)} \quad (25)$$

The canonical action functions

$$S(p,x) = \int_{t_a}^{t_b}dt\left(p\dot{x} - \frac{p^2}{2m} - \frac{1}{2}m\omega^2x^2\right) \quad (26)$$

Put Eq.26 into Eq. 25

$$K(x_b,t_b;x_a,t_a) = \prod_{n=1}^{N}\left[\int_{-\infty}^{\infty}\frac{dp_n}{2\pi\hbar}\right]\prod_{n=0}^{N}\left[\int_{-\infty}^{\infty}dx_n\right]e^{\frac{i}{\hbar}\int_{t_a}^{t_b}dt\left(p\dot{x}-\frac{p^2}{2m}-\frac{1}{2}m\omega^2x^2\right)}$$

But $p = m\dot{x}$, the path integral will be

$$K(x_b,t_b;x_a,t_a) = \prod_{n=1}^{N}\left[\int_{-\infty}^{\infty}\frac{dp_n}{2\pi\hbar}\right]\prod_{n=0}^{N}\left[\int_{-\infty}^{\infty}dx_n\right]e^{\frac{i}{\hbar}\int_{t_a}^{t_b}dt\left(\frac{1}{2}m\dot{x}^2-\frac{1}{2}m\omega^2x^2\right)}$$

Now split the path integral to classical paths x_{cl} and fluctuation paths $\delta x(t)$, we using the the fact of quadratic action function to fluctuating expand in $x = x_{cl} + \delta x$, then sum it to classical part

$$S_{cl}(x) = \int_{t_a}^{t_b}dt\left(\frac{1}{2}m\dot{x}_{cl}^2 - \frac{1}{2}m\omega^2x_{cl}^2\right)$$

And the fluctuating part,

$$S_{fl}(x) = \int_{t_a}^{t_b}dt\left(\frac{1}{2}m(\delta\dot{x})^2 - \frac{1}{2}m\omega^2(\delta x)^2\right), b.c \rightarrow \delta x(t_b) = \delta x(t_a) = 0$$

The equation of motion of SHO can be written as

$$\ddot{x}_{cl} + \omega^2x_{cl} = 0$$

The classical orbit which connection between initial and end point is

$$x_{cl}(t) = \frac{x_b\sin\omega(t-t_a) + x_a\sin\omega(t_b-t)}{\sin\omega(t_b-t_a)}, \quad for\ t_b - t_a \neq \frac{\pi}{\omega} \quad (27)$$

We can write the classical action as

$$S_{cl}(x) = \int_{t_a}^{t_b}dt\frac{1}{2}m\left(x_{cl}(-\ddot{x}_{cl} - \omega^2x_{cl})\right) + \frac{m}{2}x_{cl}\dot{x}_{cl}|_{t_a}^{t_b}$$

The first term equal to zero from equation of motion so,

$$S_{cl}(x) = \frac{m}{2}[x_{cl}(t_b)\dot{x}_{cl}(t_b) - x_{cl}(t_a)\dot{x}_{cl}(t_a)] \quad (28)$$

From Eq. 27 find $\dot{x}_{cl}(t_b) and\ \dot{x}_{cl}(t_a)$

$$\dot{x}_{cl}(t_b) = \frac{dx_{cl}}{dt}|_{t=t_b} = \frac{\omega}{\sin\omega(t_b-t_a)}(x_b\cos\omega(t_b - t_a) - x_a) \quad (29)$$

$$\dot{x}_{cl}(t_a) = \frac{dx_{cl}}{dt}|_{t=t_a} = \frac{\omega}{\sin\omega(t_b-t_a)}(x_b - x_a\cos\omega(t_b-t_a)) \quad (30)$$

(Kleinert, 2003)

Rewrite Eq. 27,29 and 30 in term of fractional paths in momentum space by using fractional Hamilton

$$H_{\alpha,\beta}(x,p) = D_\alpha|p|^\alpha + q^2|x|^\beta$$

(Laskin, 2002.)

The equation of motion of Hamilton are,

$\dot{x} = \frac{\partial H_\alpha}{\partial P}$, *and* $\dot{p} = -\frac{\partial H_\alpha}{\partial x}$

$$\dot{x}_b = \frac{\partial H_\alpha}{\partial P}\Big|_{p=p_b} = \alpha D_\alpha |p_b|^{\alpha-1}$$

$$\dot{x}_a = \frac{\partial H_\alpha}{\partial P}\Big|_{p=p_a} = \alpha D_\alpha |p_a|^{\alpha-1}$$

And,

$$\dot{x}_\alpha(t_b) = \frac{\omega \alpha D_\alpha}{\sin\omega(t_b - t_a)}\left(|p_b|^{\alpha-1}\cos\omega(t_b - t_a) - |p_a|^{\alpha-1}\right)$$

$$\dot{x}_\alpha(t_a) = \frac{\omega \alpha D_\alpha}{\sin\omega(t_b - t_a)}\left(|p_b|^{\alpha-1} - |p_a|^{\alpha-1}\cos\omega(t_b - t_a)\right)$$

Put four previous equations into Eq.28

$$S_\alpha(p) = \frac{m}{2}\frac{\omega\alpha D_\alpha}{\omega^2\sin\omega(t_b - t_a)}\left[\left((|p_b|^{2(\alpha-1)} + |p_a|^{2(\alpha-1)})\cos\omega(t_b - t_a) - 2|p_a|^{\alpha-1}|p_b|^{\alpha-1}\right)\right] \quad (31)$$

The fractional path integral of simple harmonic oscillator on momentum space can be writing as

$$K_\alpha(p_b, t_b; p_a, t_a) = \frac{1}{2\pi\hbar} e^{\frac{im}{\hbar 2}\frac{(\alpha D_\alpha)^2}{\omega\sin\omega(t_b - t_a)}\left[\left((|p_b|^{2(\alpha-1)} + |p_a|^{2(\alpha-1)})\cos\omega(t_b - t_a) - 2|p_a|^{\alpha-1}|p_b|^{\alpha-1}\right)\right]}$$

In limit $\omega \to 0$,

$$\sin\omega(t_b - t_a) = \omega(t_b - t_a),$$

$$\cos\omega(t_b - t_a) = 1$$

Then Eq. 31 will be,

$$K_\alpha(p_b, t_b; p_a, t_a) = \frac{1}{2\pi\hbar} e^{\frac{im}{\hbar 2}\frac{(\alpha D_\alpha)^2}{\omega^2(t_b - t_a)}\left[\left(|p_a|^{\alpha-1} - |p_b|^{\alpha-1}\right)^2\right]}$$

For $\alpha = 2$, back to classical case in Eq. 31

$$K_2(p_b, t_b; p_a, t_a) = \frac{1}{2\pi\hbar} e^{\frac{i}{\hbar}\frac{1}{2m\omega}\frac{1}{\sin\omega(t_b - t_a)}\left[\left((|p_b|^2 + |p_a|^2)\cos\omega(t_b - t_a) - 2p_a p_b\right)\right]}$$

Conclusion

In this paper, the fractional path integra in momentum space by time evolution method was obtained and applied to both free particles and simple harmonic oscillator, and the results were compared with the classical results when $\alpha = 2$. The fractional path integral is more general than the classical solution.

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