Fixed Point Theorem For Integral Type Inequality

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Abstract: -

Inspired and motivated by the earlier results, using the concept of weak compatibility and commutativity, we prove some common fixed point theorem for six mapping involving Ciric's type contractive condition in symmetric spaces.

Key words: - Weakly compatible maps, fixed points, symmetric spaces.

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1. INTRODUCTION:

In 2002, Branciari [3] obtained a fixed point theorem for a single mapping satisfying an analogue of Banach's contraction principle for an integral type inequality. Aliouche [2] established a common fixed point theorem for weakly compatible mappings in symmetric spaces satisfying a contractive condition of integral type and a property (E.A.) introduced by Aamri and El. Moutawakil [1]. Boikanyo and Choudhary [4] prove some common fixed point theorem for pointwise R-weakly commuting mappings in symmetric space with atleast one pair non compatible satisfying a contractive condition of integral type. They also prove some results for weakly compatible mappings.

Since then there have been many theorems dealing with mappings satisfying a general contractive condition of integral type. Some of these works are noted in B.E. Rhoades [8], Vijayaraju [11], Gairola& Rawat [5]. Recently Goyal and Jaiswal [8]obtain a common fixed point theorem by using the notion of weakly compatible mappings in symmetric space satisfying a contractive condition of integral type and a property E.A. introduced by Aamri and El. Moutawakil [1]. In this paper we obtain a common fixed point theorem by using the notion of weakly compatible mappingsinvolvingCiric's type contractive condition in symmetric spaces.

2. PRELIMINARIES:

We recall that a symmetric on a set X is a non negative real valued function d on $X\times X$ such that
(i) $d(x, y) = 0$ if and only if $x = y$,
(ii) $d(x, y) = d(y, x)$.

Let d be a symmetric on a set X and for $r > 0$ and any $x\in X$, let $B(x, r) = \{y\in X: d(x, y) < r\}$. A topology $t(d)$ on X is given by $U \in t(d)$ if and only if for each $x \in U, B(x, r)\subset U$ for some $r > 0$. A symmetric d is a semi-metric if for each $x \in X$ and each $r > 0$, $B(x, r)$ is a neighbourhood of x in the topology $t(d)$. Note that $\lim_{n\to\infty} d(x_n, x) = 0$ if and only if $x_n \to x$ in the topology $t(d)$.

The following two axioms were given by Wilson [12].
Let (X, d) be a symmetric space.
(W.3) Given $\{x_n\}$, x and y in X, $\lim_{n\to\infty} d(x_n, x) = 0$ and $\lim_{n\to\infty} d(x_n, y) = 0$ implies $x = y$.
(W.4) Given $\{x_n\}$, $\{y_n\}$ and x in X, $\lim_{n\to\infty} d(x_n, x) = 0$ and $\lim_{n\to\infty} d(x_n, y_n) = 0$ implies that $\lim_{n\to\infty} d(y_n, x) = 0$.
It is easy to see that for a semi-metric d, if $t(d)$ is a Hausdorff, then (W.3) holds.
In the sequel, we need a function $F^* = \{\varphi : R_+ \to R_+\}$ such that φ is a Lebesgue integrable mapping which is

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summable, non-negative and satisfy $\int_0^{\varepsilon} \varphi(t)dt > 0$ for all $\varepsilon > 0$ and ϕ will be a function defined by, $\phi: R_+ \to R_+$ such that $0 < \phi(t) < t$ for all $t > 0$.

Definition 1 Let S and T be two self mappings of a symmetric space (X, d). S and T are said to be compatible if $\lim_{n\to\infty} d(STx_n, TSx_n) = 0$ whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} d(Sx_n, t) = \lim_{n\to\infty} d(Tx_n, t) = 0$ for some $t \in X$.

Definition 2 Two self mappings S and T of a symmetric space (X, d) are said to be weakly compatible if they commute at their coincidence points.

Definition 3 Let S and T be two self mappings of a symmetric space (X, d). We say that S and T satisfy the property (E.A) if there exist a sequence $\{x_n\}$ such that

$$\lim_{n\to\infty} d(Sx_n, t) = \lim_{n\to\infty} d(Tx_n, t) = 0 \text{ for some } t \in X.$$

Example 1. Let $X = [0, \infty)$. Let d be a symmetric on X defined by $d(x, y) = e^{|y-x|} - 1$ for all x, y in X. Define S, T: $X \to X$ as follows:

$$Sx = 2x + 1 \text{ and } Tx = x + 2, \text{ for all } x \in X.$$

Note that the function d is not a metric. Consider the sequence $x_n = 1+1/n$, $n = 1, 2, \ldots$
Clearly $\lim_{n\to\infty} d(Sx_n, 3) = \lim_{n\to\infty} d(Tx_n, 3) = 0$.
Then S and T satisfy property (E.A), but S and T are not weakly compatible.

Definition 4 Let (X, d) be a symmetric space. We say that (X, d) satisfy property (H.E) if given $\{x_n\}$, $\{y_n\}$ and x in X, $\lim_{n\to\infty} d(x_n, x) = 0$ and $\lim_{n\to\infty} d(y_n, x) = 0$ implies $\lim_{n\to\infty} d(x_n, y_n) = 0$.

Example 2.
(i) Every metric space (X, d) satisfies property (H.E).
(ii) Let $X = [0, \infty)$ with the symmetric function d defined in Example 1. It is easy to see that the symmetric space (X, d) satisfies property (H.E).

3. MAIN RESULT

Theorem 3.1 Let d be a symmetric for X that satisfy (W.3), (W.4) and H.E. Let A, B, S, T, I and J be self mappings on (X, d) satisfying the following conditions:
(i) $I(X) \subset AB(X)$, $J(X) \subset ST(X)$,

$$\text{(ii)} \quad \int_0^{d(Ix, Jy)} \varphi(t)dt \le \phi\left(\int_0^{M(x,y)} \varphi(t)dt\right) \ldots (1)$$

for all $x, y \in X$, $\varphi \in F^*$ and

$$M(x, y) = \max\{d(STx, ABy), [d(Ix, STx)+d(Jy, ABy)], \frac{1}{2}[d(Ix, ABy)+d(Jy, STx)]\}$$

(iii) $I(X)$ or $J(X)$ is sequentially complete subspace of X.
(iv) (I, ST) and (J, AB) are weakly compatible and (I, ST) or (J, AB) satisfied the property (E.A).

Then AB, ST, I and J have a unique common fixed point.
Furthermore, if the pair (I, S), (I, T), (S, T), (J, A), (J, B) and (A, B) are commuting mappings. Then A, B, S, T, I and J have a unique common fixed point in X.

Proof: Suppose that, I and ST satisfy property (E.A.). Then there exists a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} d(Ix_n, z) = \lim_{n\to\infty} d(STx_n, z) = 0$ for some $z \in X$. Therefore, by (H.E.) $\lim_{n\to\infty} d(Ix_n, STx_n) = 0$. Since $I(X) \subset AB(X)$, there exists in X a sequence $\{y_n\}$ such that $Ix_n = ABy_n$. Hence, $\lim_{n\to\infty} d(ABy_n, z) = 0$.
Let us show that $\lim_{n\to\infty} d(Jy_n, z) = 0$.
Suppose that $\lim_{n\to\infty} \operatorname{Sup} d(Ix_n, Jy_n) > 0$. Then, using (1), we have

$$\lim_{n\to\infty} Sup \int_0^{d(Ix_n, Jy_n)} \varphi(t)dt \le \lim_{n\to\infty} Sup\,\phi\left(\int_0^{M(x_n, y_n)} \varphi(t)dt\right)$$

where $M(x_n, y_n)=\max\{d(STx_n, ABy_n), [d(Ix_n, STx_n)+d(Jy_n, ABy_n)], \frac{1}{2}[d(Ix_n, ABy_n)+d(Jy_n, STx_n)]\}$

$$= \max\{0, [0+d(Ix_n, Jy_n)], \frac{1}{2}[0 + d(Jy_n, Ix_n)]\}$$

$$\lim_{n\to\infty} Sup \int_0^{d(Ix_n, Jy_n)} \varphi(t)dt \le \lim_{n\to\infty} Sup\,\phi\left(\int_0^{d(Ix_n, Jy_n)} \varphi(t)dt\right) < \lim_{n\to\infty} Sup \int_0^{d(Ix_n, Jy_n)} \varphi(t)dt$$

which is a contradiction. Hence $\int_0^{d(Ix_n, Jy_n)} \varphi(t)dt = 0$ and $\lim_{n\to\infty} d(Ix_n, Jy_n) = 0$. By (W.4), we have $\lim_{n\to\infty} d(Jy_n, z) = 0$.

Suppose that, $I(X)$ is complete subspace of X and $I(X) \subset AB(X)$, then there exists $u \in X$ such that $ABu = z$. We have,

$$\lim_{n\to\infty} d(Jy_n, ABu)=\lim_{n\to\infty} d(Ix_n, ABu)=\lim_{n\to\infty} d(STx_n, ABu)=\lim_{n\to\infty} d(ABy_n, ABu)=0.$$

Now, we claim that $ABu = Ju$. If not, then from (1), we have

$$\int_0^{d(Ix_n, Ju)} \varphi(t)dt \le \phi\left(\int_0^{M(x_n, u)} \varphi(t)dt\right)$$

where $M(x_n, u)=\max\{d(STx_n, ABu), [d(Ix_n, STx_n)+d(Ju, ABu)], \frac{1}{2}[d(Ix_n, ABu)+d(Ju, STx_n)]\}$

$$= \max\{0, [0+d(Ix_n, Ju)], \frac{1}{2}[0 + d(Ju, Ix_n)]\}$$

$= d(Ix_n, Ju)$.

$\int_0^{d(Ix_n, Ju)} \varphi(t)dt \le \phi\left(\int_0^{d(Ix_n, Ju)} \varphi(t)dt\right)$. Letting $n\to\infty$, we obtain $\lim_{n\to\infty} \int_0^{d(Ix_n, Ju)} \varphi(t)dt = 0$, which implies $\lim_{n\to\infty} d(Ix_n, Ju) = 0$. By (W.3), we have $Ju = z = ABu$.

Using the weak compatibility of AB and J implies that $ABJu = JABu$ i.e. $ABz = Jz$. On the other hand $J(X) \subset ST(X)$, there exists $v \in X$ such that $Ju = STv$.
We claim that $STv = Jv$. If not then from (1), we have

$$\int_0^{d(STv, Iv)} \varphi(t)dt = \int_0^{d(Iv, Ju)} \varphi(t)dt \le \phi\left(\int_0^{M(v, u)} \varphi(t)dt\right)$$

where $M(v, u) = \max\{d(STv, ABu), [d(Iv, STv)+d(Ju, ABu)], \frac{1}{2}[d(Iv, ABu)+d(Ju, STv)]\}$

$$= \max\{ d(Ju, Ju), [d(Iv, Ju)+d(Ju, Ju)], \frac{1}{2}[d(Iv, Ju)+ 0]\}$$

$$= d(Iv, Ju).$$

$\int_0^{d(STv, Iv)} \varphi(t)dt \le \phi \int_0^{d(STv, Iv)} \varphi(t)dt < \int_0^{d(STv, Iv)} \varphi(t)dt$ which is a contradiction. Hence $\int_0^{d(STv, Iv)} \varphi(t)dt = 0$ which implies that $d(STv, Iv) = 0$. Then $z = Ju = ABu = STv = Iv$.

Now using the weak compatibility of ST and I implies that

$$STIv = ISTv \text{ i.e. } STz = Iz.$$

Let us show that z is a common fixed point of AB, ST, I and J.
If $z \ne Jz$, using (1), we get

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$$\int_0^{d(z,Iz)}\varphi(t)dt = \int_0^{d(Iv,Iz)}\varphi(t)dt \le \phi\left(\int_0^{M(v,z)}\varphi(t)dt\right)$$

where$M(v, z)$ = max{$d(STv, ABz)$, [$d(Iv, STv)$+$d(Jz, ABz)$], $\frac{1}{2}$ [$d(Iv, ABz)$+$d(Jz, STv)$]}

= max{$d(Iv, Jz)$, [$d(Iv, Iv)$+$d(Jz, Jz)$], $\frac{1}{2}$ [$d(Iv, Jz)$+ $d(Jz, Iv)$]}

= $d(Iv, Jz)$.

Therefore, $\int_0^{d(z,Iz)}\varphi(t)dt \le \phi\left(\int_0^{d(z,Iz)}\varphi(t)dt\right) < \int_0^{d(z,Iz)}\varphi(t)dt$, which is a contradiction.

Thus, $z = Jz = ABz$.

If $z \neq Iz$, using (1), we get

$$\int_0^{d(Iz,z)}\varphi(t)dt = \int_0^{d(Iz,Jz)}\varphi(t)dt \le \phi\left(\int_0^{M(z,z)}\varphi(t)dt\right)$$

where$M(z, z)$ = max{$d(STz, ABz)$, [$d(Iz, STz)$+$d(Jz, ABz)$], $\frac{1}{2}$ [$d(Iz, ABz)$+$d(Jz, STz)$]}

= max{$d(Iz, z)$, 0, $\frac{1}{2}$ [$d(Iz, z)$+ $d(Jz, Iz)$]}

= $d(Iz, z)$.

Therefore, $\int_0^{d(Iz,z)}\varphi(t)dt \le \phi\left(\int_0^{d(Iz,z)}\varphi(t)dt\right) < \int_0^{d(Iz,z)}\varphi(t)dt$, which is a contradiction.

Thus, $z = Iz = STz$.

Therefore, $z = Iz = STz = Jz = ABz$. i.e.z is the common fixed point of AB, ST, I and J. For the uniqueness of z, suppose that $z \neq \omega$ is another common fixed point of AB, ST, I and J. Using (1), we have

$$\int_0^{d(z,\omega)}\varphi(t)dt = \int_0^{d(Iz,J\omega)}\varphi(t)dt \le \phi\left(\int_0^{M(z,\omega)}\varphi(t)dt\right)$$

where$M(z, \omega)$ = max{$d(STz, AB\omega)$, [$d(Iz, STz)$+$d(J\omega, AB\omega)$], $\frac{1}{2}$ [$d(Iz, AB\omega)$+$d(J\omega, STz)$]}

= $d(z, \omega)$.

Therefore, $\int_0^{d(z,\omega)}\varphi(t)dt \le \phi\left(\int_0^{d(z,\omega)}\varphi(t)dt\right) < \int_0^{d(z,\omega)}\varphi(t)dt$, which is a contradiction. Therefore, $\int_0^{d(z,\omega)}\varphi(t)dt = 0$, which implies that $z = \omega$.

Now we prove that z is a common fixed point of A, B, S, T, I and J. For this let$z$is a unique common fixed point of both the pair (I, ST) and (J, AB). Using the commutativity of (I, S), (I, T) and (S, T) then

$Sz = S(STz) = S(TSz) = ST(Sz)$, $Sz = S(Iz) = I(Sz)$

and $Tz = T(STz) = TS(Tz) = ST(Tz)$, $Tz = T(Iz) = I(Tz)$

which shows that Sz and Tz are a common fixed point of (I, ST), yielding thereby

$Sz = z = Tz = Iz = STz$.

Similarly, using the commutativity of (J, A), (J, B)and(A, B) it can be shown that

$Az = z = Bz = Jz = ABz$.

Now, we need to show that $Az = Sz(Bz = Tz)$. For this let $Az \neq Sz$, using (1), we get

$$\int_0^{d(Az,Sz)}\varphi(t)dt = \int_0^{d(Sz,Az)}\varphi(t)dt = \int_0^{d(S(Iz),A(Jz))}\varphi(t)dt = \int_0^{d(I(Sz),J(Az))}\varphi(t)dt$$

$$\leq \phi\left(\int_0^{M(Sz,\,Az)} \varphi(t)dt\right)$$

where$M(Sz, Az)$ = max{d(ST(Sz), AB(Az)), [d(I(Sz), ST(Sz))+ d(J(Az), AB(Az))],

$\frac{1}{2}$ [d(I(Sz), AB(Az))+d(J(Az), ST(Sz))]}

= d(Sz, Az).

Therefore, $\int_0^{d(Az,Sz)} \varphi(t)dt \leq \phi\left(\int_0^{d(Az,Sz)} \varphi(t)dt\right) < \int_0^{d(Az,Sz)} \varphi(t)dt$, which is a contradiction. Therefore, $\int_0^{d(Az,Sz)} \varphi(t)dt = 0$ which implies that $Az = Sz$. Similarly, it can be shown that $Bz = Tz$. Thus, z is the unique common fixed point of A, B, S, T, I and J. This completes the proof.

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