Numerical Solution for a Class of Variable-order Fractional Differential Equations with Atangana-Baleanu-Caputo Fractional Derivative

H.M. Mohammadi Nejad[1*], H. Khosravi[2], O. Rabiei Motlagh[3]

Abstract

In this paper, we consider the following variable-order fractionalmobile-immobile advection-dispersion model:

$$\alpha_1 u_t(x,t) + \alpha_2 \mathfrak{D}^{ABC}_{\alpha(t)} u(x,t) = -\alpha_3 u_x(x,t) + \alpha_4 u_{xx}(x,t) + f(x,t),$$

where $0 < \alpha(t) < 1$ and $\mathfrak{D}^{ABC}_{\alpha(t)}$ denotes Atangana-Baleanu-Caputo fractional derivative of order $\alpha(t)$. We introduce Chebyshev polynomials to seek the numerical solution of the variable-order fractional mobile-immobile advection-dispersion modeland we will use of the collocation method. According to definition of Atangana-Baleanu-Caputo fractional derivative and properties of Chebyshev polynomials, fractional differential operator matrix is deduced. With the help of the operational matrixes, the equation is transformed into the products of several dependent matrixes which can also be viewed as the system of algebraic equations and by solving the linear equations.By solving linear equations, the approximate solution of the equation is calculated, and these calculations are done with the help of MATLAB software.In order to evaluate the stated method, we will perform convergence analysis and stability analysis.Among the existing techniques for investigating stability analysis, we will use the Hyers Ulam (HU) stability method.

Key Words: Variable-order Fractional, Atangana-Baleanu-Caputo Fractional Derivative, Chebyshev Polynomials.

DOI Number: 10.14704/nq.2022.20.10.NQ55408 **NeuroQuantology 2022; 20(10): 4208-4216**

Introduction

In recent decades, differential equations and their numerical solutions have been used extensively in the natural sciences, engineering, and financial mathematics. Differential equations and developing analytical and numerical methods for the solutions of fractional differential equations with variable-order fractional derivatives have essential applications in the parts of biomathematics, mathematics, chemistry, electronics, economics, engineering, etc. In the last years and also, study and discussion this type of differential equation for some modeling problems of the differential equations containing Riemann-Liouville, and Caputo derivative definitions are the most valuable tools in fractional calculus [5, 7, 10, 11, 18, 21, 25]. Recently, the numerical schemes for the solution of a class of variable-order fractional differential equations (FDEs) and the solutions of FDEs with fractional derivative have beenstudied, for example variational iteration method [16, 34], Adomian decomposition method [3, 6], generalized differential transform method [12], Wavelet Method [2], finite difference method [24, 34], collocation method [26], Expression of numerical, methods with the help of Chebyshev polynomials (CPs) [27] andcubic spline interpolation method [13] and other methods [5, 8, 14] must be used. We will discuss in this paper is given replacing Riemann-Liouville fractional integrals with Atangana-Baleanu integrals in the definition of Riemann-Liouville derivatives.

Corresponding author: H.M. Mohammadi Nejad
Address:[1*,2,3]Department of Mathematics, University of Birjand, Birjand, Iran.
E-mail: [1*]mohammadin@birjand.ac.ir

Now, we present a novel generalization of derivatives of both Riemann-Liouville and Caputo types which is obtained by modifying the Riemann-Liouville integral operator by extending its kernel with one-parameter Mittag-Leffler function, a function that extends the well-known exponential function.The mobile-immobile advection-dispersion model has been studied by different researchers with different fractional derivatives. ([36]) For example Zhang et al. [35] presented a numerical method for this model with Coimbra fractional derivative. In this paper, we seek to find the numerical solution to the following equation, which is defined as follows:

$$\alpha_1 u_t(x,t) + \alpha_2 \mathfrak{D}_{\alpha(t)}^{ABC} u(x,t) = -\alpha_3 u_x(x,t) + \alpha_4 u_{xx}(x,t) + f(x,t). \quad (1)$$

With the following initial and boundary conditions

$$u(x,0) = g(x), x \in [0,1], u(0,t) = h(t), t \in [0,1], u(x,1) = g_1(x), (\text{with consistency condition } h(0) = g(0) = c. \quad (2)$$

Where $0 < \alpha(t) < 1$ and $\alpha_1, \alpha_2 \geq 0$ and $\alpha_3, \alpha_4 > 0$ are given constants and $\boldsymbol{\mathcal{H}} \equiv [0,1] \times [0,1]$ and f and $u$are random functions of time where f, g, h and g_1 are knowncontinuous functions and u is unknown function. Here,$\mathfrak{D}_{\alpha(t)}^{ABC}$is the variable-order time fractional derivate,defined by Atangana [1]:

$$\mathfrak{D}_{\alpha(t)}^{ABC}[u(x,t)](t) = \frac{M(\alpha(t))}{1-\alpha(t)} \int_0^t \frac{\partial(u(x,\sigma))}{\partial\sigma} E_{\alpha(t)}(\frac{-\alpha(t)}{1-\alpha(t)}(t-\sigma)^{\alpha(t)}) d\sigma, \quad t > 0, u(x,t) \in \boldsymbol{H}^1(0,1). \quad (3)$$

Where $M(\alpha(t)) = 1 - \alpha(t) + \frac{\alpha(t)}{\Gamma(\alpha(t))}$and E_α the one-parametric Mittag-Leffler (M-L) function defined by the power series as:

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)}. \quad (4)$$

Where $\Gamma(.)$ is Euler Gamma function and although information on this function (M-L) is widely spread in the literature[4, 9, 17, 19]. It is easy to report the following result, namely:

(5)

Where $\mathcal{E}_{\alpha(t),m+1}$ denoted the Mittag-Leffler function with two parameters and represented as the following series[17]:

$$\mathcal{E}_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}. \quad (6)$$

Our aim in this research is to find a numerical solution for a class of variable-order FDEs using a numerical method based on the CPs matrix operators and the collocation method. By applying the proposed method, the equation (1) will become a system of algebraic equations. The rest of this paper is arranged into different sections. In section 2, we start by reminding some necessary definitions of fractional calculus. In section 3, weintroduce some essential properties of the Shifted Chebyshev polynomials (SCPs) and we will showapproximation solution. In section 4, with the help of matrix operators, we obtain the numerical solution for equation (1) and introducing the algebraic device corresponding to equation (1). In sec 5, we introduce the upper bound for absolute error and convergence analysis. Finally, in sec 6, we finish the paper by checking the uniqueness of the solution and the Hyers-Ulam stability.

Preliminaries

In this section, we introduce the mathematical fundamentals of fractional calculus, variable-order fractional calculus.

1. FractionalCalculus

In this section, first we recall the definition of the fractional integral and derivative of order $\alpha > 0$ and second we introduce the mathematical background of variable-order fractional calculus.

Definition 2.1 [4, 9, 17, 20]. Let $0 < \alpha < 1$ and $f \in L^1[a,b]$, $0 < t < b \leq \infty$. The left and right fractional integrals in the Riemann-Liouville sense of order α are defined, respectively:

$$I_{a^+}^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t f(\tau)(t-\tau)^{\alpha-1} d\tau, \quad (1)$$

$$I_{b^-}^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_t^b f(\tau)(\tau-t)^{\alpha-1} d\tau, \quad (2)$$

and also, the left and right fractional derivatives in the Riemann-Liouville sense of order α are defined as:

$$D_{a^+}^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \frac{d}{dt} \int_a^t f(\tau)(t-\tau)^{-\alpha} d\tau, \quad (3)$$

$$D_{b^-}^{\alpha} f(t) = -\frac{1}{\Gamma(\alpha)} \frac{d}{dt} \int_t^b f(\tau)(\tau-t)^{-\alpha} d\tau. \quad (4)$$

Let $f \in H^1[a,b]$. The left-sided and the right-sided Caputo fractional derivatives of order α are defined as follows.

$$ {}^{C}D_{a^+}^{\alpha}f(t)=I_{a^+}^{1-\alpha}\frac{d}{dt}f(t)=\frac{1}{\Gamma(1-\alpha)}\int_a^t(t-\tau)^{-\alpha}\frac{d}{d\tau}f(\tau)d\tau, \quad (5)$$

$$ {}^{C}D_{b^-}^{\alpha}f(t)=-I_{b^-}^{1-\alpha}\frac{d}{dt}f(t)=-\frac{1}{\Gamma(1-\alpha)}\int_t^b(\tau-t)^{-\alpha}\frac{d}{d\tau}f(\tau)d\tau. \quad (6)$$

Definition2.2[1]. For $m-1<\alpha\le m$ and $f\in L^1[0,b]$, $0<t<b\le\infty$. The Atangana-Baleanu integral of order α is defined as follows:

$$\mathbb{I}_{\alpha}^{AB}f(t)=\frac{1-\alpha}{M(\alpha)}f(t)+\frac{\alpha}{M(\alpha)}I_{a^+}^{\alpha}f(t). \quad (7)$$

Definition2.3[1]. Let $f\in L^1[0,b]$. The Atangana-Baleanu-Riemann operator for a given function f is defined as the following form:

$$\mathbb{D}_{\alpha}^{ABR}f(t)=\frac{M(\alpha)}{1-\alpha}\frac{d}{dt}\int_0^t f(\tau)E_{\alpha}\left(\frac{-\alpha}{1-\alpha}(t-\tau)^{\alpha}\right)d\tau, \quad (8)$$

Where $m-1<\alpha<m$. Let $f\in H^1[a,b]$. The Atangana-Baleanu-Caputo derivative is also defined as follows:

$$\mathbb{D}_{\alpha}^{ABC}f(t)=\frac{M(\alpha)}{1-\alpha}\int_0^t\frac{d}{d\tau}(f(\tau))E_{\alpha}\left(\frac{-\alpha}{1-\alpha}(t-\tau)^{\alpha}\right)d\tau. \quad (9)$$

2.Variable-Order Fractional Calculus

In this subsection we replace the fractional order α with a bounded function $m-1<\alpha(t)<m, m\in\mathbb{N}$ and consider several definitions for variable-order fractional derivatives and integral as follows:

Definition 2.4 [29].The Atangana-Baleanu fractional integral $\mathfrak{I}_{\alpha(t)}^{AB}$ of order α(t) is:

$$\mathfrak{I}_{\alpha(t)}^{AB}f(t)=\frac{1-\alpha(t)}{M(\alpha(t))}f(t)+\frac{\alpha(t)}{M(\alpha(t))}I_{a^+}^{\alpha(t)}f(t). \quad (10)$$

and also, The Atangana-Baleanu-Riemann fractional operator $\mathfrak{D}_{\alpha(t)}^{ABR}$ of order $\alpha(t)$ is:

$$\mathfrak{D}_{\alpha(t)}^{ABR}f(t)=\frac{M(\alpha(t))}{1-\alpha(t)}\frac{d}{dt}\int_0^t f(\tau)E_{\alpha(t)}\left(\frac{-\alpha(t)}{1-\alpha(t)}(t-\tau)^{\alpha(t)}\right)d\tau, \quad (11)$$

Definition 2.5[29]. The Atangana-Baleanu-Caputo derivative $\mathfrak{D}_{\alpha(t)}^{ABC}f(t)$of order α(t) is:

$$\mathfrak{D}_{\alpha(t)}^{ABC}f(t)=\frac{M(\alpha(t))}{1-\alpha(t)}\int_0^t\frac{d}{d\tau}(f(\tau))E_{\alpha(t)}\left(\frac{-\alpha(t)}{1-\alpha(t)}(t-\tau)^{\alpha(t)}\right)d\tau. \quad (12)$$

Shifted Chebyshev Polynomials (SCPs)

The well-known Chebyshev polynomials have been very successfully used in many scientific and engineering fields and can be determined on the interval $x\in[-1,1]$ with the following recurrence formula [21, 31-33]:

$$T_{n+1}(x)=2xT_n(x)-T_{n-1}(x), n=1,2,3,\cdots,\quad T_0(x)=1, T_1(x)=x. \quad (13)$$

Analytically, we have,

$$T_n(x)=n\sum_{i=0}^{\left[\frac{n}{2}\right]}(-1)^i2^{n-2i-1}\frac{(n-i-1)!}{(i)!\,(n-2i)!}x^{n-2i}. \quad (14)$$

Chebyshev polynomials functions have the following normality and orthogonality properties:

$$\int_{-1}^1 T_i(x)T_j(x)(1-x^2)^{\frac{-1}{2}}dx=\begin{cases}\pi, & i=j=0,\\ \frac{\pi}{2}, & i=j\neq 0,\\ 0 & j\neq i.\end{cases} \quad (15)$$

The shifted Chebyshev polynomials on the interval [0,1] can be defined as [21, 31-33]:

$$T_{m+1}^*(t)=2(2t-1)T_m^*(t)-T_{m-1}^*(t), m=1,2,3,\dots,\quad T_0^*(t)=1, T_1^*(t)=2t-1. \quad (16)$$

Analytically, we have

$$T_m^*(t)=m\sum_{k=0}^{m}(-1)^{m-k}2^{2k}\frac{(m+k-1)!}{(2k)!\,(m-k)!}t^k. \quad (17)$$

They satisfy the following orthogonality condition:

$$\int_0^1 T_i^*(x)T_j^*(x)\mathfrak{S}_x dx=\begin{cases}\pi, & i=j=0,\\ \frac{\pi}{2}, & i=j\neq 0,\\ 0 & j\neq i,\end{cases} \quad (18)$$

where $\mathfrak{S}_x$=$(x-x^2)^{\frac{-1}{2}}$ is the weight function. Let

$$\Phi(t)=[T_0^*(t),T_1^*(t),\dots,T_n^*(t)]^T, \quad (19)$$

We can define the shifted Chebyshev vector as a matrix form as follows:

$$\Phi(t)=A\boldsymbol{T_n}(\boldsymbol{t}), \quad (20)$$

Where $A, \mathcal{T}_n$ are defined by:

$$A = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ -1 & 2 & 0 & \dots & 0 \\ 2(-1)^2 \frac{1!}{2!} & 2(-1)^1 \frac{2^2 2!}{2!} & 2(-1)^0 \frac{2^4 3!}{4!} & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ n(-1)^n \frac{(n-1)!}{n!} & n(-1)^{n-1} \frac{2^2 (n)!}{2!\,(n-1)!} & n(-1)^{n-2} \frac{2^4 (n+1)!}{4!\,(n-2)!} & \dots & n(-1)^0 \frac{2^{2n}(2n-1)!}{(2n)!} \end{bmatrix} \quad (21)$$

$$\mathcal{T}_n(t) = \begin{bmatrix} 1 \\ t \\ \vdots \\ t^n \end{bmatrix}. \quad (22)$$

It is obvious that:

$$\mathcal{T}_n(t) = A^{-1}\Phi(t). \quad (23)$$

A function $u(t) \in L^2(0,1)$ can be expanded by shifted Chebyshev polynomials as follows:

$$u(t) = \sum_{i=0}^{\infty} c_i T_i^*(t). \quad (24)$$

By truncating the infinity series given in Eq. (24), can be approximated as follows:

$$u(t) \cong \sum_{i=0}^{n} c_i T_i^*(t) = c^T \Phi(t), \quad (25)$$

We can obtain an approximate expression of $u(x,t) \in L^2([0,1] \times [0,1])$ as follows:

$$u(x,t) \cong \sum_{i=0}^{n} \sum_{j=0}^{n} u_{ij} T_i^*(x) T_j^*(t) = \Phi^T(x) U \Phi(t), \quad (26)$$

Where

$$U = (u_{ij}), 0 \le i,j \le n, n \epsilon \mathbb{N}. \quad (27)$$

The matrix U is unknown of size $(n+1)\times(n+1)$. The components u_{ij} are determined by MATLAB software (or with least square method if necessary).

Transformation of Differential Operators

We write the differential operators of the integer order and fraction in the form of a matrix.Using differentiation of vector Φ in the relation (7), then we have:

$$\Phi'(t) = D\Phi(t) = D(A\mathcal{T}_n(t)) = AD(\mathcal{T}_n(t)) = AD\left(\begin{bmatrix} 1 \\ t \\ \vdots \\ t^n \end{bmatrix}\right) = A \begin{bmatrix} 0 \\ 1 \\ \vdots \\ nt^{n-1} \end{bmatrix} = AV\mathcal{T}_n^*(t), \quad (28)$$

Where$V, \mathcal{T}_n^*(t)$ can be shown as follows:

$$V = \begin{bmatrix} 0 & 0 & \dots & 0 \\ 1 & 0 & \dots & 0 \\ 0 & 2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & n \end{bmatrix}, \quad (29)$$

$$\mathcal{T}_n^*(t) = \begin{bmatrix} 1 \\ t \\ \vdots \\ t^{n-1} \end{bmatrix}_{n\times 1}. \quad (30)$$

Therefore, we obtain

$$\mathcal{T}_n^*(t) = B^*\Phi(t), \quad (31)$$

Where

$$B^* = \begin{bmatrix} A_{[1]}^{-1} \\ A_{[2]}^{-1} \\ \vdots \\ A_{[n]}^{-1} \end{bmatrix}, \quad (32)$$

and $A_{[k]}^{-1}$ is the k −th row of $A^{-1}, k = 1,2,3, \dots n$. Therefore, we obtain

$$\Phi'(t) = AVB^*\Phi(t). \quad (33)$$

The shape of the operational matrix for u_t, u_x *and* u_{xx} is as follows:

$$u_t(x,t) \approx \Phi^T(x) UAVB^* \Phi(t). \quad (34)$$

$$u_x(x,t) \approx \frac{\partial(\Phi^T(x)U\Phi(t))}{\partial x} = \mathrm{D}(\Phi^T(x))U\Phi(t) = \Phi^T(x)(AVB^*)^T U\Phi(t), \quad (35)$$

eISSN 1303-5150

$$u_{xx}(x,t)=\frac{\partial\left(u_x(x,t)\right)}{\partial x}\approx\frac{\partial\left(\Phi^T(x)(AVB^*)^TU\Phi(t)\right)}{\partial x}=\Phi^T(x)((AVB^*)^T)^2U\Phi(t). \tag{36}$$

Theorem 4.1. Let $u(x,t)\in L^2([0,1]\times[0,1])$ and $0<\alpha(t)<1$. Then we have

$$\mathfrak{D}^{ABC}_{\alpha(t)}[u(x,t)]=\Phi^T(x)UA\mathrm{N}A^{-1}\Phi(t), \tag{37}$$

$$\mathrm{N}=\begin{bmatrix}\frac{\Gamma(1)M(\alpha(t))}{1-\alpha(t)}\mathcal{E}_{\alpha(t),1}(-\frac{\alpha(t)t^{\alpha(t)}}{1-\alpha(t)}) & \cdots & 0\\ \vdots & \ddots & \vdots\\ 0 & \cdots & \frac{\Gamma(n+1)M(\alpha(t))}{1-\alpha(t)}\mathcal{E}_{\alpha(t),n+1}(-\frac{\alpha(t)t^{\alpha(t)}}{1-\alpha(t)})\end{bmatrix}$$

Proof. With the help of relation (26) we have

$$\mathfrak{D}^{ABC}_{\alpha(t)}[u(x,t)]=\mathfrak{D}^{ABC}_{\alpha(t)}[\Phi^T(x)U\Phi(t)]=\Phi^T(x)U\mathfrak{D}^{ABC}_{\alpha(t)}[\Phi(t)]=\Phi^T(x)U\mathfrak{D}^{ABC}_{\alpha(t)}[[T_0^*(t),T_1^*(t),\dots,T_n^*(t)]^T]$$
$$=\Phi^T(x)U[\mathfrak{D}^{ABC}_{\alpha(t)}(T_0^*(t)),\mathfrak{D}^{ABC}_{\alpha(t)}(T_1^*(t)),\dots,\mathfrak{D}^{ABC}_{\alpha(t)}(T_n^*(t))]^T$$
$$=\Phi^T(x)U\left[\mathfrak{D}^{ABC}_{\alpha(t)}(1),\mathfrak{D}^{ABC}_{\alpha(t)}(2t-1),\dots,\mathfrak{D}^{ABC}_{\alpha(t)}\left(n\sum_{k=0}^{n}\left((-1)^{n-k}2^{2k}\frac{(n+k-1)!}{(2k)!\,(n-k)!}t^k\right)\right)\right]^T$$
$$=\Phi^T(x)U\begin{bmatrix}1 & 0 & 0 & \dots & 0\\ -1 & 2 & 0 & \dots & 0\\ 2(-1)^2\frac{1!}{2!} & 2(-1)^1\frac{2^2 2!}{2!} & 2(-1)^0\frac{2^4 3!}{4!} & \dots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ n(-1)^n\frac{(n-1)!}{n!} & n(-1)^{n-1}\frac{2^2(n)!}{2!\,(n-1)!} & n(-1)^{n-2}\frac{2^4(n+1)!}{4!\,(n-2)!} & \dots & n(-1)^0\frac{2^{2n}(2n-1)!}{(2n)!}\end{bmatrix}$$
$$\times\begin{bmatrix}\frac{\Gamma(1)M(\alpha(t))}{1-\alpha(t)}\mathcal{E}_{\alpha(t),1}\left(-\frac{\alpha(t)t^{\alpha(t)}}{1-\alpha(t)}\right) & \cdots & 0\\ \vdots & \ddots & \vdots\\ 0 & \cdots & \frac{\Gamma(n+1)M(\alpha(t))}{1-\alpha(t)}\mathcal{E}_{\alpha(t),n+1}\left(-\frac{\alpha(t)t^{\alpha(t)}}{1-\alpha(t)}\right)\end{bmatrix}\times\begin{bmatrix}1\\ t\\ \vdots\\ t^n\end{bmatrix}$$
$$=\Phi^T(x)UA\mathrm{N}_{(n+1)(n+1)}A^{-1}\Phi(t).$$

Now by merging the obtained relations, equations (1)-(2) becomes the following form:

$$\begin{gathered}\alpha_1\Phi^T(x)UAVB^*\Phi(t)+\alpha_2\Phi^T(x)UA\mathrm{N}_{(n+1)(n+1)}A^{-1}\Phi(t)=-\Phi^T(x)(AVB^*\Phi(x))U\Phi(t)+\alpha_4\Phi^T(x)((AVB^*\Phi(x))^T)^2U\Phi(t)+f(x,t),\\ u(x,.)\approx\Phi^T(x)U\Phi(0)=g(x),u(0,t)\approx\Phi^T(0)U\Phi(t)=h(t),u(x,1)\approx\Phi^T(x)U\Phi(1)=g_1(x).\end{gathered} \tag{38}$$

We define the residual function as follows:

$$\mathrm{R}(x,t)=\alpha_1\Phi^T(x)UAVB^*\Phi(t)+\alpha_2\Phi^T(x)UA\mathrm{N}_{(n+1)(n+1)}A^{-1}\Phi(t)+\Phi^T(x)\left(AVB^*\Phi(x)\right)U\Phi(t)-\alpha_4\Phi^T(x)((AVB^*\Phi(x))^T)^2U\Phi(t)-f(x,t).$$

Eventually, the following system with $(n+1)\times(n+1)$ equations can be generated from Eq.(38):

$$\begin{cases}\mathrm{R}(x_i,t_j)=0, i=1,2,\dots,n-1, j=1,2,\dots,n\\ g(x_i)=0, i=0,1,2,\dots,n\\ h(t_j)=0, g_1(x_i)=0, j=1,2,\dots,n,\end{cases}$$

where $x_i=\frac{1}{2}(1+\cos(\frac{2i-1}{2n}\pi))$, $t_j=\frac{1}{2}(1+\cos(\frac{2j-1}{2n}\pi))$,
i, j=0, 1, 2, …, n.

Obviously, by solving the above device, the approximate solution of equation (1) is obtained.

Convergence Analysis and Error Assessment

Theorem 5.1. Assume that u_E (exact solution) be a smooth function on $\mathcal{H}$. Also, let $\Upsilon_n=\mathrm{Span}\{T_i^*(x)T_j^*(t)|\, i,j\in\aleph=\{0,1,\dots,n\}\}$ and u_I

is the interpolating polynomial for u_E at the points $(x_i,\ t_j)$, where $x_i,(0 \le i \le n)$ are the roots of $T^*_{n+1}(x)$,while where t_j, $(0 \le i \le n)$ are the roots of $T^*_{n+1}(t)$and u_B is the best approximation for u_E. Therefore, the following relationship is established.

$$\|u_E - u_B\|_{L^2(\mathcal{H})} \xrightarrow{n\to\infty} 0 \quad (39)$$

Proof. Since $u_B \epsilon \mathbf{\Upsilon_n}$ is the best approximation for u_E,then

$$\|u_E - u_B\|_{L^2(\mathcal{H})} \le \|u_E - u^*\|_{L^2(\mathcal{H})}, \forall u^* \epsilon \mathbf{\Upsilon_n}. \quad (40)$$

We know that the previous inequality also holds for u_I (interpolating polynomial for u_E).According [28], there exist $\sigma, \varsigma, \sigma' \, and\ \varsigma' \epsilon[0,1]$, such that

$$\begin{aligned}|u_E - u_I| \le [&\frac{1}{(n+1)!}|\Pi_{i=0}^n(x-x_i)||u_{E(x,n+1)}(\sigma,t)| + \frac{1}{(n+1)!}|\Pi_{i=0}^n(t-t_i)||u_{E(t,n)}(x,\varsigma)| \\ &+ \frac{1}{(n+1)!}\frac{1}{(n+1)!}|\Pi_{i=0}^n(t-t_i)| |\Pi_{i=0}^n(x-x_i)| \left|u_{E((t,n+1),(x,n+1))}(\sigma',\varsigma')\right|] \\ \le [&\frac{1}{(n+1)!}Max|\Pi_{i=0}^n(x-x_i)|Max|u_{E(x,n+1)}(\sigma,t)| \\ &+ \frac{1}{(n+1)!}Max|\Pi_{i=0}^n(t-t_i)|Max|u_{E(t,n)}(x,\varsigma)| + \frac{1}{(n+1)!}\frac{1}{(n+1)!}Max|\Pi_{i=0}^n(t-t_i)| \\ &Max|\Pi_{i=0}^n(x-x_i)|Max\left|u_{E((t,n+1),(x,n+1))}(\sigma',\varsigma')\right|],\end{aligned} \quad (41)$$

where $u_{E(x,n)} \equiv \frac{\partial^n u_E}{\partial x^n}, u_{(t,n)} \equiv \frac{\partial^n u_E}{\partial t^n}, u_{((t,n),(x,m))} \equiv \frac{\partial^{n+m} u_E}{\partial x^n \partial t^m}$, $(x,t)\in \mathcal{H}$.According to the assumption, there exist three real-valued constants $\lambda_1, \lambda_2, \lambda_3 >.$, such that

$$Max\left|u_{E\,(x,n)}(\sigma,t)\right| \le \lambda_1, Max\left|u_{E\,(t,n)}(x,\varsigma)\right| \le \lambda_2, Max\left|u_{E\,(t,n)}(x,\varsigma)\right| \le \lambda_3.$$

Let λ=max{$\lambda_1, \lambda_2,\ \lambda_3$}. Also, we obtain

$$\underbrace{Min}_{x_i\in[0,1]}\ \underbrace{\text{Max}}_{x\in[0,1]}|\Pi_{i=0}^n(x-x_i)| = \underbrace{Min}_{s_i\in[0,1]}\ \underbrace{\text{Max}}_{s\in[0,1]}\left|\Pi_{i=0}^n\frac{1}{2}(s-s_i)\right| = (\frac{1}{2})^{n+1}|\Pi_{i=0}^n(s-s_i)| = \frac{1}{4^n},$$

where $s_i s$ are the root of $T_{n+1}(s)$.Therefore we obtain

$$\begin{aligned}|u_E - u_I| &\le \frac{1}{(n+1)!}\frac{1}{4^n}\lambda + \frac{1}{(n+1)!}\frac{1}{4^n}\lambda + (\frac{1}{(n+1)!})^2\frac{1}{16^n}\lambda \\ &= \lambda[(\frac{1}{(n+1)!})(\frac{1}{4^n}) + \left(\frac{1}{(n+1)!}\right)^2(\frac{4}{16^n})] \equiv \boldsymbol{\mathcal{Q}}(\lambda,\text{n})\end{aligned} \quad (42)$$

Eventually, Eqs. (40)-(42) results in

$$\|u_E - u_B\|_{L^2(\mathcal{H})} \le \left\|u_E - u_I\right\|_{L^2(\mathcal{H})} = \left[\int_0^1\int_0^1|u_E - u_I|^2\varpi(\text{x})\,\varpi(\text{t})\text{dxdt}\right]^{\frac{1}{2}} \le \left[\int_0^1\int_0^1|\boldsymbol{\mathcal{Q}}(\lambda,\text{n})|^2\varpi(\text{x})\,\varpi(\text{t})\text{dxdt}\right]^{\frac{1}{2}} =$$

$\boldsymbol{\mathcal{Q}}$λ, n$2\pi 2$≡**M(**λ, n, π).

HU-Stability Analysis

Stability analysis is an important part of numerical methods. We run Hyers Ulam-stability analysis of the given Eq.(1). Let $\alpha_2 = 1$. For this purpose, by catching Atangana-Baleanu-fractional integral operator to the Eq. (1), we have

$$u(x,t) = u(x,0) + \frac{1-\alpha(t)}{M(\alpha(t))}[\alpha_4 u_{xx}(x,t) - \alpha_3 u_x(x,t) - \alpha_1 u_t(x,t) + f(x,t)] + \frac{\alpha(t)}{M(\alpha(t))\Gamma(\alpha(t))}\int_0^t (t-m)^{\alpha(t)-1}[\alpha_4 u_{xx}(x,m) - \alpha_3 u_x(x,m) - \alpha_1 u_t(x,m) f(x,m)]dm. \quad (43)$$

For simplicity taking

$$\mathbb{L}(x,t,u) = \alpha_4 u_{xx} - \alpha_3 u_x - \alpha_1 u_t + f, \quad \frac{1-\alpha(t)}{M(\alpha(t))} = \boldsymbol{\mathcal{F}(t)}. \quad (44)$$

Then, Eq. (43) gives

$$u(x,t) = u(x,0) + \boldsymbol{\mathcal{F}(t)}\mathbb{L}(x,t,u) + \frac{\alpha(t)}{M(\alpha(t))\Gamma(\alpha(t))}\int_0^t (t-m)^{\alpha(t)-1}\mathbb{L}(x,m,u)dm. \quad (45)$$

Notice.$(\mathfrak{D}_1)$ Let $u_i = u_i(x,t), u_i^* = u_i^*(x,t)$ and $u, u^* \in L^2([0,1] \times$ 0,1. We assume that three positive constants δi, ρsuch that the following inequalities hold true:

$|u_i - u_i^*| \leq \delta_i$,

$$|u_{xx} - u_{xx}^*| \leq \rho.$$

Notice. The kernel $\mathbb{L}$ is satisfying the following inequality provided that $(\mathfrak{D}_1)$ holds true.

$$|\mathbb{L}(x,t,u) - \mathbb{L}(x,t,u^*)| = |\alpha_4(u_{xx}(x,t) - u_{xx}^*(x,t)) - \alpha_3(u_x(x,t) - u_x^*(x,t)) - \alpha_1(u_t(x,t) - u_t^*(x,t))|$$
$$\leq \alpha_4|(u_{xx}(x,t) - u_{xx}^*(x,t)| + \alpha_3|(u_x(x,t) - u_x^*(x,t)| + \alpha_1 u_t(x,t) - u_t^*(x,t)u_t(x,t) - u_t^*(x,t)|$$
$$\leq (\alpha_4\rho + \alpha_3\delta_x + \alpha_1\delta_t)|u(x,t) - u^*(x,t)|.(46)$$

Theorem 6.1. Assume that $(\mathfrak{D}_1)$ holds true. Then, the model (1) has a unique solution provided that the following holds true:

$$[\frac{1-\alpha(t)}{M(\alpha(t))} + \frac{1}{M(\alpha(t))\Gamma(\alpha(t))}](\alpha_4\rho + \alpha_3\delta_x + \alpha_1\delta_t) \leq 1.$$

Proof. We suppose that there exists another solution like$\hat{u}$of the model (1), such that the integral system given by (46) is satisfied. Then, we obtain

$$\hat{u}(x,t) = \hat{u}(x,0) + \frac{1-\alpha(t)}{M(\alpha(t))}\mathbb{L}(x,t,\hat{u}) + \frac{\alpha(t)}{M(\alpha(t))\Gamma(\alpha(t))}\int_0^t (t-\tau)^{\alpha(t)-1}\mathbb{L}(x,\tau,\hat{u})d\tau.(48)$$

We obtain

$$|u(x,t) - \hat{u}(x,t)| \leq [\frac{1-\alpha(t)}{M(\alpha(t))} + \frac{1}{M(\alpha(t))\Gamma(\alpha(t))}](\alpha_4\rho + \alpha_3\delta_x + \alpha_1\delta_t) \| u(x,t) - \hat{u}(x,t) \|.$$

That shows

$$[1 - (\left[\frac{1-\alpha(t)}{M(\alpha(t))} + \frac{1}{M(\alpha(t))\Gamma(\alpha(t))}\right](\alpha_4\rho + \alpha_3\delta_x + \alpha 1\delta t)]|u(x,t) - u(x,t)| \leq 0.$$

Now, according to the assumption of the theorem, we have

$$|u(x,t) - \hat{u}(x,t)| \leq 0.$$

This result proves that the answer of the model (1) is unique.

Definition 6.2[30].The Eq. (45) is said to be HU-stable if there exists$\gamma_2 > 0$such that, for every $\gamma_1 > 0$, for

$$|u(x,t) - u(x,0) - \boldsymbol{\mathcal{F}(t)}\mathbb{L}(x,t,u) - \frac{\alpha(t)}{M(\alpha(t))\Gamma(\alpha(t))}\int_0^t (t-m)^{\alpha(t)-1}\mathbb{L}(x,m,u)dm| < \gamma_1, \quad (49)$$

there exists a unique solution$u^*(x,t)$ such that

$$u^*(x,t) - u^*(x,0) - \boldsymbol{\mathcal{F}(t)}\mathbb{L}(x,t,u^*) - \frac{\alpha(t)}{M(\alpha(t))\Gamma(\alpha(t))}\int_0^t (t-m)^{\alpha(t)-1}\mathbb{L}(x,m,u^*)dm, \quad (50)$$

implies

$$|u - u^*| < \gamma_1\gamma_2. \quad (51)$$

For the HU-stability of the given Eq. (1) by the help of definition 6.1, the assumed Eq. (1) has a unique solution say u.

Let u and u^*be approximate and exact solution respectively, of the given Eq. (1). Then, we obtain

$$|u - u^*| \leq \boldsymbol{\mathcal{F}(t)}|\mathbb{L}(x,t,u) - \mathbb{L}(x,t,u^*)| + \frac{\alpha(t)}{M(\alpha(t))\Gamma(\alpha(t))}\int_0^t (t-m)^{\alpha(t)-1}|\mathbb{L}(x,m,u) - \mathbb{L}(x,m,u^*)|dm \leq [\boldsymbol{\mathcal{F}(t)} + \frac{1}{M(\alpha(t))\Gamma(\alpha(t))}]|\mathbb{L}(x,t,u) - \mathbb{L}(x,t,u^*)|. \quad (52)$$

From Eq. (44), we obtain

$$|\mathbb{L}(x,t,u) - \mathbb{L}(x,t,u^*)| \leq (\alpha_4\rho + \alpha_3\delta_x + \alpha_1\delta_t)|u - u^*|. \quad (53)$$

Therefore, we obtain

$$|u - u^*| \leq [\boldsymbol{\mathcal{F}(t)} + \frac{1}{M(\alpha(t))\Gamma(\alpha(t))}](\alpha_4\rho + \alpha_3\theta_x + \alpha_1\theta_t) \| u - u^* \| = \gamma_1\gamma_2. \quad (54)$$

Conclusion

In recent decades, many researchers have expanded their studies to different areas of fractional calculus. It seems obvious that in most models resulting from a natural phenomenon, finding the analytical answer of the model is either not possible or very difficult. For this reason, numerical methods have been used extensively by researchers. One of the famous natural models is the mobile-immobile advection-dispersion model. In this paper, the numerical solution of this model is based on the collocation method and the use of operational matrices including CPs. These operational matrices transformed equation (1) into an algebraic device. In the following, with the help of Lagrange interpolation polynomials, we introduced an upper bound for the absolute error and checked the convergence analysis. Then we assumed that the exact solution of the equation is smooth, and we assumed that the solution of the equation exists. Also, we checked its uniqueness, and finally, according to the structure of the equation and the properties of the equation, among the stability analysis techniques, we used Hyers-Ulam's stability technique.

References

Atangana A, Baleanu D. New fractional derivatives with nonlocal and non-singular kernel: theory and application to heat transfer model. arXiv preprint arXiv:1602.03408. 2016 Jan 20.

Chen Y, Yi M, Yu C. Error analysis for numerical solution of fractional differential equation by Haar wavelets method. Journal of Computational Science, 2012 Sep 1;3(5):367-73.

El-Kalla IL. Convergence of the Adomian method applied to a class of nonlinear integral equations. Applied mathematics letters, 2008 Apr 1;21(4):372-6.

Garra R, Gorenflo R, Polito F, Tomovski Ž. Hilfer–Prabhakar derivatives and some applications. Applied mathematics and computation, 2014 Sep 1;242:576-89.

Han W, Chen YM, Liu DY, Li XL, Boutat D. Numerical solution for a class of multi-order fractional differential equations with error correction and convergence analysis. Advances in Difference Equations., 2018 Dec;2018(1):1-22.

Hosseini MM. Adomian decomposition method for solution of nonlinear differential algebraic equations. Applied mathematics and computation, 2006 Oct 15;181(2):1737-44.

Ichise M, Nagayanagi Y, Kojima T. An analog simulation of non-integer order transfer functions for analysis of electrode processes. Journal of Electroanalytical Chemistry and Interfacial Electrochemistry, 1971 Dec 1;33(2):253-65.

Jafari H, Tajadodi H, Ganji RM. A numerical approach for solving variable order differential equations based on Bernstein polynomials. Computational and Mathematical Methods, 2019 Sep;1(5):e1055.

Kilbas AA, Srivastava HM, Trujillo JJ. Theory and applications of fractional differential equations. Elsevier; 2006 Mar 2.

Khubalkar S, Junghare A, Aware M, Das S. Unique fractional calculus engineering laboratory for learning and research. The International Journal of Electrical Engineering & Education, 2020 Jan;57(1):3-23.

Lai J, Mao S, Qiu J, Fan H, Zhang Q, Hu Z, Chen J. Investigation progresses and applications of fractional derivative model in geotechnical engineering. Mathematical Problems in Engineering, 2016 May 4;2016.

Momani S, Odibat Z, Erturk VS. Generalized differential transform method for solving a space-and time-fractional diffusion-wave equation. Physics Letters A. 2007 Oct 29;370(5-6):379-87.

Moghaddam BP, Machado JA. A computational approach for the solution of a class of variable-order fractional integro-differential equations with weakly singular kernels. Fractional Calculus and Applied Analysis, 2017 Aug 1;20(4):1023-42.

Najafi HS, Edalatpanah SA, Refahi Sheikhani AH. Convergence analysis of modified iterative methods to solve linear systems. Mediterranean journal of mathematics, 2014 Aug;11(3):1019-32.

Narumi S. Some formulas in the theory of interpolation of many independent variables. Tohoku Mathematical Journal, First Series. 1920;18:309-21.

Odibat ZM. A study on the convergence of variational iteration method. Mathematical and Computer Modelling. 2010 May 1;51(9-10):1181-92.

Podlubny, I. Fractional differential equations: an introduction to fractional derivatives, fractional differential equations, to methods of their solution and some of their applications, 1998 (Vol. 198). Elsevier.

Ortigueira MD. Fractional calculus for scientists and engineers. Springer Science & Business Media; 2011 Jun 2.

Prabhakar TR. A singular integral equation with a generalized Mittag Leffler function in the kernel.

Srivastava HM, Saxena RK, Pogany TK, Saxena R. Integral and computational representations of the extended Hurwitz–Lerch zeta function. Integral Transforms and Special Functions, 2011 Jul 1;22(7):487-506.

Sun K, Zhu M. Numerical algorithm to solve a class of variable order fractional integral-differential equation based on Chebyshev polynomials. Mathematical Problems in Engineering, 2015 Jan 1;2015.

Sun H, Zhang Y, Baleanu D, Chen W, Chen Y. A new collection of real world applications of fractional calculus in science and engineering. Communications in Nonlinear Science and Numerical Simulation. 2018 Nov 1;64:213-31.

Snyder, M. A. Snyder M. Chebyshev Method in Numerical Analysis. 1996. (Vol. 2). Prentice-Hall.

Xu Y, Suat Ertürk V. A finite difference technique for solving variable-order fractional integro-differential equations. Bulletin of the Iranian Mathematical Society. 2014 Jun 1;40(3):699-712.

Xu Z, Chen W. A fractional-order model on new experiments of linear viscoelastic creep of Hami Melon. Computers & Mathematics with Applications, 2013 Sep 1;66(5):677-81.

Zayernouri M, Karniadakis GE. Fractional spectral collocation methods for linear and nonlinear variable order FPDEs. Journal of Computational Physics, 2015 Jul 15;293:312-38.

Mohammadinejad H, Khosravi H. Numerical solution for a class of variable order fractional integral-differential equation with Atangana-Baleanu-Caputo fractional derivative. Journal of Advanced Mathematical Modeling, 2021 Jun 22;11(2):253-70.

De Villiers J. Mathematics of approximation. Springer Science & Business Media; 2012 Jun 30.

Ávalos-Ruiz LF, Zúñiga-Aguilar CJ, Gómez-Aguilar JF, Escobar-Jiménez RF, Romero-Ugalde HM. FPGA implementation and control of chaotic systems involving the variable-order fractional operator with Mittag–Leffler law. Chaos, Solitons & Fractals, 2018 Oct 1;115:177-89.

Huang J, Li Y. Hyers–Ulam stability of linear functional differential equations. Journal of Mathematical Analysis and Applications, 2015 Jun 15;426(2):1192-200.

Nejla G, Emine GK, Yusuf G. Chebyshev-Tau method for the linear Klein-Gordon equation. International Journal of Physical Sciences, 2012 Nov 16;7(43):5723-8.

Ji T, Hou J, Yang C. Numerical solution of the Bagley–Torvik equation using shifted Chebyshev operational matrix. Advances in Difference Equations, 2020 Dec;2020(1):1-4.

Mahdy A, Shwayyea R. Shifted Chebyshev polynomials of the third kind solution for system of non-linear fractional diffusion equations. International Journal of Advance Research, 2016;4:1-20.

Chen Y, Liu L, Li B, Sun Y. Numerical solution for thevariable order linear cable equation with Bernsteinpolynomials. Applied Mathematics and Computation, 2014 Jul 1;238:329-41.

Zhang H, Liu F, Phanikumar MS, Meerschaert MM. A novel numerical method for the time variable fractional order mobile–immobile advection–dispersion model. Computers & Mathematics with Applications, 2013 Sep 1;66(5):693-701.

Liu F, Zhuang P, Burrage K. Numerical methods and analysis for a class of fractional advection–dispersion models. Computers & Mathematics with Applications, 2012 Nov 1;64(10):2990-3007.

Liu F, Zhuang P, Burrage K. Numerical methodsand analysis for a class of fractional advection–dispersionmodels. Computers & Mathematics with Applications, 2012 Nov 1;64(10):2990-3007.