



ON CYCLIC AMENABILITY OF TRIANGULAR BANACH ALGEBRAS

MOHAMMAD HOSSAIN REZEI GOL¹ AND EBRAHIM NASRABADI^{2, *}

650

ABSTRACT. The inherited properties between the triangular Banach algebra in the form of $T = \begin{bmatrix} A & M \\ 0 & B \end{bmatrix}$ and the corner Banach algebras A and B are very important and studied. In this paper, first a method for investigating being cyclic of derivations on Banach algebras is presented and then the cyclic derivations on triangular Banach algebras is studied. Using the above method, we show that the unital triangular Banach algebra T is cyclic amenable if and only if the corner algebras A and B are cyclic amenable. Finally, an example shows that the condition of unitary of T is necessary for the correctness of the results.

1. INTRODUCTION

The concept of an amenable Banach algebra was defined and studied for the first time by Johnson in [7]. After that, the concept of weak amenability for Banach algebras was studied by Bade along with Curtis and Dales in [1] and examined weak amenability for some Banach algebras. The concept of cyclic amenability was presented by Gronbeak in [5]. He investigated the hereditary properties of this concept. Of course, each weak amenable Banach algebra is cyclic amenable, but the converse is not necessarily true.

On the other hand, Forrest and Marcoux in [3] have studied the structure of derivations on triangular Banach algebras in the form of

$$T = \left\{ \begin{bmatrix} a & m \\ 0 & b \end{bmatrix}; a \in A, b \in B, m \in M \right\}$$

equipped with the usual operations associated with 2×2 matrices and the Banach space norm $\left\| \begin{bmatrix} a & m \\ 0 & b \end{bmatrix} \right\| = \|a\| + \|m\| + \|b\|$, where in A and B are Banach algebras, and M be a Banach A ; B -module, that means, M is a left Banach A -module and right Banach B -module. They have studied the weak amenability of triangular Banach algebras in [4] and show that, the weak amenability of T is equivalent to the weak amenability of the corner Banach algebras A and B , when T is unital. The properties of triangular Banach algebras have been discussed in many papers and the inherited properties between the triangular Banach algebra $T = \begin{bmatrix} A & M \\ 0 & B \end{bmatrix}$ and the corner Banach algebras A and B are very important and studied. Given the recent progress in studying derivations on triangular Banach algebra, it is natural to focus on studying certain types of derivatives (for example, cyclic derivations) on these algebras.

From where the concept of cyclic amenability is weaker than the concept of weak amenability, the question is, between T and its corner Banach algebras, the cyclic amenability is also inherited?

Like what happened to the weak amenability in the Forrest and Morcoux's results. In this paper, we show that the answer to this question is yes. One of the useful features of our results is that we can create new cyclic amenability Banach algebras with different structures.

algebras A and B are very important and studied. Given the recent progress in studying derivations on triangular Banach algebra, it is natural to focus on studying certain types of derivatives (for example, cyclic derivations) on these algebras.

¹ 2010 Mathematics Subject Classification. Primary 46H20; Secondary 46H25, 16E40.

Key words and phrases. triangular Banach algebra, cyclic derivation, cyclic amenability.

EISSN 1303-5150



We first present a method for investigating being cyclic of derivations on unital Banach algebras and then we study the structure of the cyclic derivation on triangular Banach algebras T . Next, we show that T is cyclic amenable if and only if the corner Banach algebras A and B are cyclic amenable.

Let A be a Banach algebra and X be a Banach A -bimodule. A bounded (continuous) linear map $D: A \rightarrow X$ is called a derivation if

$$D(ab) = a \cdot D(b) + D(a) \cdot b \quad (a, b \in A).$$

Every $x \in X$ defines a derivation $\text{ad}_x(a) = a \cdot x - x \cdot a$ ($a \in A$), which is called inner derivation. We use the notations $Z^1(A, X)$ and $B^1(A, X)$ for the set of all derivations from A in to X , and the set of all inner derivations from A in to X , respectively. The first cohomology group of A with

coefficients in X are defined to be

$$H^1(A, X) = Z^1(A, X)/B^1(A, X).$$

Let A be a Banach algebras. We know that doul of A , which is displayed by A^* , is a Banach A -bimodule, with the following action

$$(a \cdot f)(b) = f(b \cdot a) \quad \text{and} \quad (f \cdot a)(b) = f(a \cdot b) \quad (a, b \in A, f \in A^*).$$

Definition 1. Banach algebra A is called amenable if for every Banach A -bimodule X , every derivation $D: A \rightarrow X^*$ is inner, or $H^1(A, X^*) = 0$. A is called weak amenable if $H^1(A, A^*) = 0$.

Definition 2. Derivation $D: A \rightarrow A^*$ is called cyclic if $[D(a)](b) + [D(b)](a) = 0$, for every $a, b \in A$.

We use the notation $ZC^1(A, A^*)$ for the set of all cyclic derivations from A into A^* , and the first cyclic cohomology group of A is defined to be

$$HC^1(A, A^*) = ZC^1(A, A^*)/B^1(A, A^*).$$

Definition 3. Banach algebra A is called cyclic amenable if every cyclic derivation $D: A \rightarrow A^*$, is inner, or $HC^1(A, A^*) = \{0\}$.

The next lemma provide method for investigating being cyclic of derivations on unital Banach algebras, which is useful to us and we will use it several times in our current work.

Lemma 1.1. Let A be an unital Banach algebra with unit 1_A and $D: A \rightarrow A^*$ be a derivation. Then D is cyclic if and only if $[D(a)](1_A) = 0$, for each $a \in A$.

Proof. Let D be a derivation. For every $a, b \in A$,

$$\begin{aligned} [D(a)](b) + [D(b)](a) &= [D(a)](b1_A) + [D(b)](1_A a) \\ &= [D(a) \cdot b](1_A) + [a \cdot D(b)](1_A) \\ &= [D(a) \cdot b + a \cdot D(b)](1_A) \\ &= [D(ab)](1_A). \end{aligned}$$

This shows that, D is cyclic if and only if $[D(c)](1_A) = 0$, for each $c \in A^2$. But $A^2 = A$, because A is unital and the proof is complete.

2. CYCLIC DERIVATIONS ON TRIANGULAR BANACH ALGEBRAS

Let A and B be Banach algebras and M be a Banach A, B -module, that means, M is a left Banach A -module and right Banach B -module. Consider the previously introduced triangular Banach algebra as follows

$$T = \text{Tri}(A, M, B) = \left\{ \begin{bmatrix} a & m \\ 0 & b \end{bmatrix}; a \in A, m \in M, b \in B \right\}$$



If A and B are unital with units $\mathbf{1}_A$ and $\mathbf{1}_B$, respectively, and M is unital, that means, $\mathbf{1}_A \cdot m = m \cdot \mathbf{1}_B = m$ for each $m \in M$, then T is also unital with unit $\mathbf{1}_T = \begin{bmatrix} \mathbf{1}_A & \\ & \mathbf{1}_B \end{bmatrix}$.

Conversely, if T is unital with unit $\mathbf{1}_T$, Then A and B are also unital with units $\mathbf{1}_A = \pi_1(\mathbf{1}_T)$ and $\mathbf{1}_B = \pi_4(\mathbf{1}_T)$, respectively, where π_1 and π_4 are projection maps on corner arrays.

The triangular Banach algebra T , as a Banach space, is isomorphic to the l^1 -direct sum $A \oplus_1 M \oplus_1 B$. Therefore, $T^* \simeq A^* \oplus_\infty M^* \oplus_\infty B^*$, with the pairing

$$\varphi(t) = f(a) + g(m) + h(b) \quad \left(t = \begin{bmatrix} a & m \\ & b \end{bmatrix} \in T, \varphi = \begin{bmatrix} f & g \\ & h \end{bmatrix} \in T^* \right) \quad (2,1)$$

It follows immediately that the module actions of T on T^* are as follows:

$$\begin{aligned} \begin{bmatrix} a & m \\ & b \end{bmatrix} \cdot \begin{bmatrix} f & g \\ & h \end{bmatrix} &= \begin{bmatrix} a \cdot f + m \cdot g & b \cdot g \\ & b \cdot h \end{bmatrix}, \\ \begin{bmatrix} f & g \\ & h \end{bmatrix} \cdot \begin{bmatrix} a & m \\ & b \end{bmatrix} &= \begin{bmatrix} f \cdot a & g \cdot a \\ g \cdot m + h \cdot b \end{bmatrix}. \end{aligned} \quad (2,2)$$

for every $\begin{bmatrix} a & m \\ & b \end{bmatrix} \in T$ and $\begin{bmatrix} f & g \\ & h \end{bmatrix} \in T^*$.

Note that in the previous relation, $m \cdot g$ and $g \cdot m$ correctly belongs to A^* and B^* , respectively, by the rules $(m \cdot g)(a) = g(a \cdot m)$ and $(g \cdot m)(b) = g(m \cdot b)$, where $m \in M, g \in M^*, a \in A$ and $b \in B$.

Lemma 2.1. [3, Theorem 2.1] Let T be a unital triangular Banach algebra, $D: T \rightarrow T^*$ be a derivation and define $D_A: A \rightarrow A^*$ and $D_B: B \rightarrow B^*$ by

$$D_A(a) = \pi_1(D\left(\begin{bmatrix} a & 0 \\ & 0 \end{bmatrix}\right)) \text{ and } D_B(b) = \pi_4(D\left(\begin{bmatrix} 0 & 0 \\ & b \end{bmatrix}\right)). \quad (2.3)$$

Then D_A and D_B are derivations and there exist $\gamma \in M^*$ such that

$$D\left(\begin{bmatrix} a & m \\ & b \end{bmatrix}\right) = \begin{bmatrix} D_A(a) - m\gamma & \gamma a - b\gamma \\ & D_B(b) + \gamma m \end{bmatrix} \quad (2.4)$$

Lemma 2.2. Let T be a unital triangular Banach algebra, $D: T \rightarrow T^*$ be a derivation and D_A and D_B be derivations defined by (2.3). Then

- (i) D is cyclic if and only if both D_A and D_B are cyclic.
- (ii) D is inner if and only if both D_A and D_B are inner.

Proof. (i) Let D be cyclic. So $[D(t)](\mathbf{1}_T) = 0$ for every $t \in T$, by lemma 1.1. For every

$a \in A$ we have

$$[D_A(a)](\mathbf{1}_a) = \begin{bmatrix} D_A(a) & \gamma a \\ & 0 \end{bmatrix} \begin{bmatrix} \mathbf{1}_A & 0 \\ & \mathbf{1}_B \end{bmatrix} = \left[D\left(\begin{bmatrix} a & 0 \\ & 0 \end{bmatrix}\right) \right] \begin{bmatrix} \mathbf{1}_A & 0 \\ & \mathbf{1}_B \end{bmatrix} = 0$$

and similarly for every $b \in B$,

$$[D_B(b)](\mathbf{1}_b) = \begin{bmatrix} 0 & -b\gamma \\ & D_B(b) \end{bmatrix} \begin{bmatrix} \mathbf{1}_A & 0 \\ & \mathbf{1}_B \end{bmatrix} = \left[D\left(\begin{bmatrix} 0 & 0 \\ & b \end{bmatrix}\right) \right] \begin{bmatrix} \mathbf{1}_A & 0 \\ & \mathbf{1}_B \end{bmatrix} = 0$$

Due to lemma 1.1 we deduce that that D_A and D_B are cyclic.



Conversely, let D_A and D_B be cyclic derivations. By using (2.1) and (2.4), For each $t = \begin{bmatrix} a & m \\ b & \end{bmatrix} \in T$ we have

$$\begin{aligned} [D(t)](1_T) &= \left[D \left(\begin{bmatrix} a & m \\ b & \end{bmatrix} \right) \right] \begin{pmatrix} 1_A & 0 \\ & 1_B \end{pmatrix} \\ &= \begin{bmatrix} D_A(a) - m\gamma & \gamma a - b\gamma \\ D_B(b) + \gamma m & \end{bmatrix} \begin{pmatrix} 1_A & 0 \\ & 1_B \end{pmatrix} \\ &= [D_A(a) - m\gamma] (1_A) + [D_B(b) + \gamma m] (1_B) \\ &= [D_A(a)] (1_A) - [m\gamma] (1_A) + [D_B(b)] (1_B) + [\gamma m] (1_B) \\ &= 0 - \gamma(1_A m) + 0 + \gamma(m 1_B) = 0. \end{aligned}$$

Therefore, D is cyclic by reusing lemma 1.1.

(ii) Because of the communication between D , D_A and D_B in (2.4) and using (2.2) we have

$$\begin{aligned} D \left(\begin{bmatrix} a & m \\ b & \end{bmatrix} \right) &= \begin{bmatrix} D_A(a) - m\gamma & \gamma a - b\gamma \\ D_B(b) + \gamma m & \end{bmatrix} \\ &= \begin{bmatrix} D_A(a) \\ D_B(b) \end{bmatrix} + \begin{bmatrix} -m\gamma & -b\gamma \\ 0 & \end{bmatrix} + \begin{bmatrix} 0 & \gamma a \\ \gamma m & \end{bmatrix} \\ &= \begin{bmatrix} D_A(a) \\ D_B(b) \end{bmatrix} - ad \begin{bmatrix} 0 & \gamma \\ & 0 \end{bmatrix} \left(\begin{bmatrix} a & m \\ b & \end{bmatrix} \right) \end{aligned}$$

Therefore, we can see that the observation that

$$\begin{bmatrix} D_A(a) \\ D_B(b) \end{bmatrix} = \left[D + ad \begin{bmatrix} 0 & \gamma \\ & 0 \end{bmatrix} \right] \left(\begin{bmatrix} a & m \\ b & \end{bmatrix} \right) \left(\begin{bmatrix} a & m \\ b & \end{bmatrix} \in T \right) \quad (2.5)$$

Now let D be inner. So, $D = \mathbf{ad}_\omega$, for some $\omega = \begin{bmatrix} f & g \\ & h \end{bmatrix}$. The relation (2.5) by using (2.2), for $m = 0$ and any $a \in A$ and $b \in B$, shows that

$$\begin{aligned} \begin{bmatrix} D_A(a) \\ D_B(b) \end{bmatrix} &= \left[D + ad \begin{bmatrix} 0 & \gamma \\ & 0 \end{bmatrix} \right] \left(\begin{bmatrix} a & 0 \\ b & \end{bmatrix} \right) \\ &= ad \begin{bmatrix} f & g+\gamma \\ & h \end{bmatrix} \left(\begin{bmatrix} a & 0 \\ b & \end{bmatrix} \right) \\ &= \left(\begin{bmatrix} a & 0 \\ b & \end{bmatrix} \right) \begin{bmatrix} f & g+\gamma \\ & h \end{bmatrix} - \begin{bmatrix} f & g+\gamma \\ & h \end{bmatrix} \left(\begin{bmatrix} a & 0 \\ b & \end{bmatrix} \right) \\ &= \begin{bmatrix} af & b(g+\gamma) \\ & bh \end{bmatrix} - \begin{bmatrix} fa & (g+\gamma)a \\ & hb \end{bmatrix} \\ &= \begin{bmatrix} ad_f(a) & b(g+\gamma) - (g+\gamma)a \\ & ad_h(b) \end{bmatrix} \end{aligned}$$

Therefore, $D_A = \mathbf{ad}_f$ and $D_B = \mathbf{ad}_h$ and so D_A and D_B are inner. Similarly, we can show that D is inner if D_A and D_B are inner. Indeed, if $D_A = \mathbf{ad}_f$ and $D_B = \mathbf{ad}_h$ for some $f \in A^*$ and $h \in B^*$, it is easy to check $D = \mathbf{ad}_\omega$, where $\omega = \begin{bmatrix} f & -\gamma \\ & h \end{bmatrix}$



3. CYCLIC AMENABILITY OF TRIANGULAR BANACH ALGEBRAS

In this section, we show that the triangular Banach algebra T is cyclic amenable if and only if the corner Banach algebras A and B are cyclic amenable, in the case of unitary.

Theorem 3.1. *Let $T = \text{Tri}(A, M, B)$ be a unital triangular Banach algebra. Then*

$$HC^1(T, T^*) \simeq HC^1(A, A^*) \oplus HC^1(B, B^*).$$

Proof. Let $D \in ZC^1(T, T^*)$. By lemma 2.2, there exist cyclic derivations $D_A \in ZC^1(A, A^*)$ and $D_B \in ZC^1(B, B^*)$ and $\gamma \in M^*$ such that (2.4) is valid.

Consider the map

$$\begin{aligned} \Gamma: ZC^1(T, T^*) &\rightarrow HC^1(A, A^*) \oplus HC^1(B, B^*) \\ D &\mapsto (D_A + B^1(A, A^*), D_B + B^1(B, B^*)). \end{aligned}$$

Part (i) of lemma 2.2 guarantees that Γ is well define. Clearly Γ is additive. Let $D \in ZC^1(T, T^*)$ and $\lambda \in \mathbb{C}$. Since projection maps on arrays are linear, for every $a \in A$ we have

$$[\lambda D]A(a) = \left[\pi_1 \left(\lambda D \left(\begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} \right) \right) \right] = \left[\lambda \pi_1 \left(D \left(\begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} \right) \right) \right] = \lambda D_A(a)$$

This shows that $[\lambda D]_A = \lambda D_A$. Similarly $[\lambda D]_B = \lambda D_B$. Therefore, clearly Γ is linear. By part

(ii) of lemma 2.2, D is inner derivation if and only if both D_A and D_B are inner derivations. This means that $\text{Ker } \Gamma = B^1(T, T^*)$.

Finally let $P: A \rightarrow A^*$ and $Q: B \rightarrow B^*$ be cyclic derivations on A and B , respectively.

Define $R: T \rightarrow T^*$ by

$$R \left(\begin{bmatrix} a & m \\ 0 & b \end{bmatrix} \right) = \begin{bmatrix} p_A(a) & 0 \\ 0 & Q(b) \end{bmatrix}$$

It's easy to check R is derivation. Now for every $\begin{bmatrix} a & m \\ 0 & b \end{bmatrix} \in T$, we have

$$\begin{aligned} \left[R \left(\begin{bmatrix} a & m \\ 0 & b \end{bmatrix} \right) \right] (1_T) &= \begin{bmatrix} p_A(a) & 0 \\ 0 & Q(b) \end{bmatrix} \left(\begin{bmatrix} 1_A & 0 \\ 0 & 1_B \end{bmatrix} \right) \\ &= [P(a)] (1_A) + [Q(b)] (1_B) \\ &= 0. \end{aligned}$$

Hence, R is a cyclic derivation. Obviously, $R_A = P$ and $R_B = Q$ and so Γ is surjective. Elementary linear algebra theory now implies that

$$HC^1(T, T^*) = \frac{ZC^1(T, T^*)}{B^1(T, T^*)} = \frac{ZC^1(T, T^*)}{\text{ker } \Gamma} \simeq \text{Im } \Gamma = HC^1(A, A^*) \oplus HC^1(B, B^*)$$

Corollary 3.1.1. *Let $T = \text{Tri}(A, M, B)$ be a unital triangular Banach algebra. Then T is cyclic amenable if and only if both A and B are cyclic amenable.*

Remark 1. *In proof of theorem 3.1, the map Γ is surjective, even if where A , B and M are not unital. So, we always have $HC^1(T, T^*) \supseteq HC^1(A, A^*) \oplus HC^1(B, B^*)$. That's mean, if $HC^1(T, T^*) = 0$ then $HC^1(A, A^*) = HC^1(B, B^*) = 0$.*

Now by previous remark, we have the following corollary.

Corollary 3.1.2. *Let A and B be Banach algebras and $A \oplus_i B$ be cyclic amenable. Then A and B are cyclic amenable. The converse is true when A and B are unital.*



In the general case, the converse of corollary 3.1.2 is not true. Indeed, the condition unitary A, B is necessary as we will see in the next example.

Example 1. Let A be a non-zero Banach algebra with zero product, that is $ab = 0$ for all $a, b \in A$. It is shown in [5, Example 2.5] that such A is cyclic amenable if and only if its dimension is one. This imply that C with the zero product is cyclic amenable, while $T = \text{Tri}(C, 0, C) = C \oplus_l^1 C$ is not cyclic amenable, because its dimension is more than one.

Example 2. Let $A = C \oplus_l^1 C$ where C with the zero product. By Example 1, since A is not cyclic amenable, so the triangular Banach algebra $\begin{bmatrix} A & A \\ & A \end{bmatrix}$ and $\begin{bmatrix} A & \\ & A \end{bmatrix}$ are also not cyclic amenable by Remark 1 and corollary 3.1.2 (ii), respectively.

REFERENCES

- [1] BADE, W. G. —CURTIS, P. C. —DALES, H. G.: *Amenability and weak amenability for Beurling and Lipschitz algebras*, Proc. London. Math. Soc., **3**(1987), No. 2, 359-377.
- [2] ESSLAMZAGEH, G. H. —SHOJAEE, B.: *Approximate weak amenability of Banach algebras*, Bull. Belg. Math. Soc., **18**(2011), 415-429.
- [3] FORREST, B. E. — MARCOUX, L. W.: *Derivation of triangular Banach algebras*, Indiana, Univ. Math. Soc. J., **45** (1996), 441-462.
- [4] FORREST, B. E. — MARCOUX, L. W.: *Weak amenability of triangular Banach algebras*, Trans. Amer. Math. Soc. **354** (2002), 1435-1452.
- [5] GRØNBÆK N.: *Weak and cyclic amenability for non-commutative Banach algebras*, Proc. Edinburgh. Math. Soc., **35** (1992), 315-328.
- [6] GHAHRAMANI, F. —LOY, R. J.: *Generalized notions of amenability*, J. Funct. Anal., **208** (2004), 229-260.
- [7] JOHNSON, B. E.: *Cohomology in Banach algebras*, Memoirs Amer. Math. Soc. **127** (1972).
- [8] SHOJAEE, B. —BOGAGHI, A.: *A generalization of cyclic amenability of Banach algebras*, Mathematica Slovaca. **65** (2015), No. 3, 633-644.

¹ FACULTY OF MATHEMATICS SCIENCE AND STATISTICS, UNIVERSITY OF BIRJAND, P.O. BOX 414 BIRJAND 9717851367, IRAN
 E-mail address: rezaei@birjand.ac.ir

² FACULTY OF MATHEMATICS SCIENCE AND STATISTICS, UNIVERSITY OF BIRJAND, P.O. BOX 414 BIRJAND 9717851367, IRAN
 E-mail address: nasrabadi@birjand.ac.ir

