On Common Fixed Point in G-metric Space Using (CLR_F) Property.

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Received: 30 October 2022, Accepted: 27 December 2022

Abstract: In this paper, we stablish common fixed point theorem in G-metric space using (CLR_F) property. Our result generalized the result of Zead Mustafa, Hassen Aydi and Erdal Karapnar (*Mustafa et al.,* 2012).
Keywords: Metric Space, G-Metric Space, Common Fixed Point, Weakly Commuting, Weakly Compatible Mapping, (E.A) Property, (CLR_F) Property.
Subject Classification: 54H25, 47H10.

DOI Number: 10.48047/nq.2022.20.22.NQ10051 **NeuroQuantology2022;20(22): 702-709**

Introduction

Metric fixed point plays a crucial part in matrices because of its wide range of applications in practical mathematics and science. The concept of the Banach contraction, one of the most significant discoveries in fixed point theory, has been generalized in a large number of ways.

In 1976, Jungck (*Jungck*, 1976) provided proof of the fixed point theorem for commuting maps. However, in order for his conclusion to be valid, the map in question needs to be continuous. A watered-down version of the commutativity principle for self-maps was provided by Sessa **(***Sessa*, 1982). Later on, Jungck (*Jungck,* 1986) improved his result by introducing the idea of compatible mapping in order to extend the assumption that weak commuting maps are compatible. In 1996, Jungck (*Jungck*, 1994) provided a definition for a more general idea referred to as weakly compatible maps.

Commuting⇒Weakly Commuting⇒Compatible⇒Weakly Compatible is our one way connotation.

The concept of property (E. A), which includes the class of non-compatible mapping, was defined by Amari Moutawakil in 2002 (*Aamri et al.,* 2002). Imdad and Ali (*Imdad et a*l., 2008) used the (E.A) property to demonstrate the common fixed point theorem. The concept of (CLRg) property was defined by Sintunavarat and Kumam (*Sintunavarat et al.,* 2011). It has been made aware that the (CLR_F) attribute never requires the subspace to be complete (or close).

Using the (CLR_F) property, we demonstrate Zead Mustafa, Hassen Aydi and Erdal Karapnar (*Mustafa et al.,* 2012) results in this study. Our findings do not require completion (or proximity) of subspace to demonstrate the existence of a common fixed point. By utilizing the notification described in this paper, many of the popular fixed point theorems in the body of current literature can be demonstrated.

Preliminaries:

Definition (2.1) (*Aamri et al.,* 2002): Let T and F be two self-mapping of a metric space (X, d). If there exists a sequence (x_n) such that

$$\lim_{n\to\infty} T\,x_n = \lim_{n\to\infty} F\,x_n = t,$$

then T and F satisfy the (E.A) property for some $t \in X$.

Definition (2.2) (*Diwan et al.,* 2016): Let T and F be two self-mapping of a metric space (X, d). We say that T and F satisfy the common limit range $_F$(CLR_F) property if there exists a sequence (x_n) such that

$$\lim_{n\to\infty} T\,x_n = \lim_{n\to\infty} F\,x_n = Fu,$$

for some $u \in X$.

Example (2.1): Let $X = [0, \infty)$ with the usual matric on X. Define $T, F: X \to X$ by $Tx = \frac{x}{3}$ and $Fx = 3x$ for all $x \in X$. Consider the sequence $\{x_n\} = \left\{\frac{1}{n}\right\}$. Since $\lim_{n\to\infty} T\,x_n = \lim_{n\to\infty} I\,x_n = 0 = F0$, therefore, T and F satisfy the (CLR_F) property.

Definition (2.3) (*Sessa,* 1982): In a metric space (X, d), two self-mapping T and F are said to be weakly commuting if

$$d(TFx, FTx) \le d(Tx, Fx), \quad \forall\, x \in X.$$

It is clear that two commuting mapping are weakly commuting but converse is not true.

Definition (2.4) (***Jungck*, 1986**)**:** In a metric space (X, d), two self-mapping T and F are the said to be compatible if

$$\lim_{n\to\infty} d(TFx_n, FTx_n) = 0,$$

whenever $\{x_n\}$ is a sequence in X such that

$$\lim_{n\to\infty} T\,x_n = \lim_{n\to\infty} F\,x_n = t,$$

for some $t \in X$.

Definition (2.5): In a metric space (X, d), two self-mapping T and F are said to be weakly compatible if they commute at their coincidence points, i.e. if $Tu = Fu$ for some $u \in X$, than $TFu = FTu$.

It is easy to see that two compatible mapping are weakly compatible.

Definition (2.6) (*Mustafa et al.,* 2012): A G-metric space is a pair (X, G), where X is a nonempty set and G is a non-negative real-valued function defined on $X \times X \times X$ such that for all $x, y, z, a \in X$ we have

$(G1)$ $G(x, y, z) = 0$ if $x = y = z$;

$(G2)$ $0 < G(x, x, y)$; for all $x, y \in X$, with $x \neq y$;

$(G3)$ $G(x, x, y) \leq G(x, y, z)$ for all $x, y, z \in X$, with $z \neq y$;

$(G4)$ $G(x, y, z) = G(x, z, y) = G(y, z, x) = \cdots \dots$, (symmetry in all three variable); and

$(G5)$ $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$ for all $x, y, z, a \in X$, (Rectangle inequality).

The function G is called a G-metric on X.

Every G-metric on X defines a metric d_G on X by

$$d_G(x, y) = G(x, y, y) + G(y, x, x) \quad \text{for all } x, y \in X.$$

Example (2.2): Let (X, d) be a metric space. The function $G: X \times X \times X \to [0, +\infty)$, defined by

$$G(u, v, w) = max\,\{d(u, v), d(v, w), d(w, u)\},$$

or

$$G(u, v, w) = d(u, v) + d(v, w) + d(w, u),$$

for all $u, v, w \in X$, is a G-metric on X.

Definition (2.7) (*Mustafa et al.,* 2006): A sequence (x_n) in a G-metric space X is said to converge if there exists $x \in X$, such that $\lim_{n,m \to \infty} G(x, x_n, x_m) = 0$, and one say that the (x_n) is G-convergent to x and we call x the limit of the sequence (x_n) and write $x_n \to x$ or $\lim_{n \to \infty} x_n \to x$.

Proposition (2.1) (*Mustafa et al.,* 2006): Let X be a G-metric space. than the following statement are equivalent

(1) (x_n) is convergent to x.

(2) $G(x_n, x_n, x) \to 0,\ as\ n \to \infty$.

(3) $G(x_n, x, x) \to 0,\ as\ n \to \infty$.

(4) $G(x_m, x_n, x) \to 0, as\ m, n \to \infty$.

Definition (2.8) (*Mustafa et al.,* 2006): In a G-metric space X, a sequence (x_n) is said to be G-Cauchy if given $\varepsilon > 0$, there is $N \in$ N such that $G(x_n, x_m, x_l) < \varepsilon$, for all $n, m, l \geq N$, that is $G(x_n, x_m, x_l) \to 0$ as $n, m, l \to \infty$.

Proposition (2.2) (*Mustafa et al.,* 2006): In a G-metric space X, then the following statement are equivalent

(1) The sequence (x_n) is G-Cauchy.

(2) For every $\varepsilon > 0$, there exists $N \in$ N such that $G(x_n, x_m, x_m) < \varepsilon$, for all $n, m \geq N$.

Proposition (2.3) (*Mustafa et al.,* 2006): Let X be a G-metric space, then the function $G(x, y, z)$ is jointly continuous in all three of its variable.

Definition (2.9) (*Mustafa et al.,* 2006): A G-metric space X is said to be complete if every G-Cauchy sequence in X is G- convergent in X.

Proposition (2.4) (*Abbas et al.,* 2009): Let T and F be a pair of self-maps in X that are weakly compatible. If T and F share a unique point of coincidence, $w = Tx = Fx$, then w is their unique common fixed point.

Now, we rewrite Definition on G-metric spaces.

Definition (2.10) (*Mustafa et al.,* 2012): A pair of self-mapping (T, F) of a G-metric space (X, G) is said to be G-weakly commuting of type G_T if

$$G(TFx, FTx, TTx) \leq G(Tx, Fx, Tx), \quad \text{for all } x \in X. \tag{2.1}$$

Definition (2.11) (*Mustafa et al.,* 2012): A pair of self-mapping (T, F) of a G-metric space (X, G) is said to be G-R weakly commuting of type G_T if there exists some positive real number R such that

$$G(TFx,\ FTx,\ TTx) \leq RG(Tx,\ Fx,\ Tx), \quad \text{for all } x \in X. \tag{2.2}$$

Remark (2.1): The G-weakly commuting maps of type G_T are G-R weakly commuting of type G_T. Reciprocally, if $R \leq 1$, then G-R weakly commuting maps of type G_T are G-weakly commuting of type G_T.

If we interchange T and F in (2.1) and. (2.2) then the pair of mapping (T, F) is called G-weakly commuting of type G_T and G-R weakly commuting of type G_T, respectively.

Definition (2.12) (*Mustafa et al.,* 2012): Let T and F represent a pair of self-maps in a G-metric space (X, G). If there is a sequence (x_n) such that (Tx_n) and (Ix_n) G-converge to the u for some $u \in X$, then we can say that T and F satisfy the (E.A) property. (From Proposition 2.1).

$$\lim_{n\to\infty} G\left(Tx_n, Ix_n, t\right) = \lim_{n\to\infty} G\left(Ix_n, Ix_n, t\right) = 0.$$

Main Results

Definition (3.1): Let T and F represent a pair of self-maps in a G-metric space (X, G)**.** If there is a sequence (x_n) such that (Tx_n) and (Fx_n) G-converge to the u for some $u \in X$, then we can say that T and F satisfy the (CLR_F) property (From Proposition 2.1)

$$\lim_{n\to\infty} G\left(Tx_n, Tx_n, Fu\right) = \lim_{n\to\infty} G\left(Fx_n, Fx_n, Fu\right) = 0.$$

Following to Matkowski (*Matkowski,* 1977), let $\boldsymbol{\Phi}$ be the set of all function ϕ such that $\phi : [0, \infty) \to [0, \infty)$ be a non decreasing function with $\lim_{n\to\infty} \phi^n(t) = 0$ for all $t \in (0, +\infty)$. If $\phi \in \Phi$, then ϕ is called a $\Phi -$ map. If ϕ *is* a $\Phi -$ map, then it is easy to show that

(1) $\phi(t) < t$ for all $t \in (0, +\infty)$.

(2) $\phi(0) = 0$.

Theorem (3.1): Let T and F be two self-mappings of a G-metric space (X, G) that are weakly compatible, such that

(1) T and F satisfy the (CLR$_F$) property,

(2) $$G\left(T(x), T(y), T(z)\right) \leq \phi\left(max\begin{Bmatrix} G\big(F(x), F(y), F(z)\big), & G\big(F(x), T(x), F(z)\big), \\ G\big(F(z), T(z), F(z)\big), & G\big(F(y), T(y), F(z)\big) \end{Bmatrix}\right), \quad (3.1)$$

for all $x, y, z \in X$, then T and F have unique common fixed point.

Proof: Since T and F satisfy the (CLR$_F$) property, there exists a sequence (x_n) in X such that

$$\lim_{n\to\infty} T\, x_n = \lim_{n\to\infty} F\, x_n = F(u),$$

for some $u \in X$.

First we show that $T(u) = F(u)$. Suppose that $T(u) \neq F(u)$ then $d(Tu, Fu) > 0$. Using condition (2) with $x = u$, $y = u$ and $z = x_n$, then we get

$$G\big(T(u), T(u), T(x_n)\big) \leq \phi\left(max\begin{Bmatrix} G\big(F(u), F(u), F(x_n)\big) \\ G\big(F(u), T(u), F(x_n)\big) \\ G\big(F(x_n), T(x_n), F(x_n)\big) \end{Bmatrix}\right). \quad (3.2)$$

Taking the limit as $n \to \infty$ we get

$$G\big(T(u), T(u), I(u)\big) \leq \phi\left(max\begin{Bmatrix} G\big(F(u), F(u), F(u)\big) \\ G\big(F(u), T(u), F(u)\big) \\ G\big(F(u), T(u), F(u)\big) \end{Bmatrix}\right), \quad (3.3)$$

$$G\big(T(u), T(u), F(u)\big) \leq \phi\left(max\begin{Bmatrix} G\big(F(u), F(u), F(u)\big) \\ G\big(F(u), T(u), F(u)\big) \end{Bmatrix}\right)$$
$$= \phi\Big(G\big(F(u), T(u), F(u)\big)\Big).$$

Therefore

$$G(T(u), T(u), F(u)) \leq \phi\Big(G\big(F(u), T(u), F(u)\big)\Big)$$
$$< \Big(G\big(F(u), T(u), F(u)\big)\Big). \quad (3.4)$$

Similarly

$$G\big(F(u), F(u), T(u)\big) < G\big(F(u), T(u), T(u)\big). \quad (3.5)$$

Hence from (3.4) and (3.5), we get

$$G\big(T(u), T(u), F(u)\big) < G\big(F(u), T(u), T(u)\big),$$

a contradiction, hence $T(u) = F(u)$.

Since T and F are weakly compatible, then $FT(u) = TF(u)$ and therefore,

$$TT(u) = TF(u) = FT(u) = FF(u).$$

Finally we show that $T(u)$ is common fixed point of T and F. Suppose that $T(u) \neq TT(u)$. Then using condition $x = T(u)$, $y = u$ and $z = u$ we get

$$G\left(TT(u), T(u), T(u)\right) \leq \phi\left(max\begin{Bmatrix} G\big(FT(u), F(u), F(u)\big), & G\big(FT(u), TT(u), F(u)\big), \\ G\big(F(u), T(u), F(u)\big), & G\big(F(u), T(u), F(u)\big) \end{Bmatrix}\right). \quad (3.6)$$

Putting $T(u) = F(u)$ we will get

$$G\left(TT(u),\ T(u),T(u)\right) \leq \phi\left(max\begin{Bmatrix} G(TT(u),T(u),T(u)) \\ G(TT(u),TT(u),T(u)) \\ G(T(u),T(u),T(u)) \end{Bmatrix}\right)$$

$$= \phi\left(max\begin{Bmatrix} G(TT(u),T(u),T(u)) \\ G(TT(u),TT(u),T(u)) \end{Bmatrix}\right)$$

$$< max\begin{Bmatrix} G(TT(u),T(u),T(u)) \\ G(TT(u),TT(u),T(u)) \end{Bmatrix}.$$

Thus

$$G(TT(u),T(u),T(u)) < G(TT(u),TT(u),T(u)). \tag{3.7}$$

Similarly

$$G(TT(u)\,TT(u),T(u)) < G(TT(u),T(u),T(u)). \tag{3.8}$$

It is a contradiction so

$$T(u) = TT(u) \text{ and } FT(u) = TT(u) = T(u).$$

Thus $T(u)$ is common fixed point of mapping T and F.

Uniqueness of the common fixed point is direct consequences of condition (2).

Example (3.1): Let $X = (0,1]$ and $G(x,y,z) = \max(|x-y|, |y-z|, |x-z|)$ for all $x,y,z \in X$.

Defined the mapping $T, F : X \to X$ by

$$T(x) = \begin{cases} \frac{2}{3}, & 0 < x \leq \frac{2}{3} \\ \frac{1}{3}, & \frac{2}{3} < x \leq 1. \end{cases},$$

$$F(x) = \begin{cases} 1 - \frac{x}{2}, & 0 < x \leq \frac{2}{3}, \\ \frac{4}{5}, & \frac{2}{3} < x \leq 1. \end{cases}$$

It is clear that T and F satisfy (CLR$_F$) property. To see this let us consider sequence $x_n = \frac{2}{3} - \frac{1}{n}$.

Then $\lim_{n\to\infty} T x_n = \lim_{n\to\infty} F x_n = \frac{2}{3} = F\left(\frac{2}{3}\right)$

Also $T\left(\frac{2}{3}\right) = F\left(\frac{2}{3}\right) \Longrightarrow TF\left(\frac{2}{3}\right) = FT\left(\frac{2}{3}\right).$

Which shows that T and F weakly compatible. On the other hand, a simple calculation gives that

$$G(Tx,Ty,Tz) \leq \phi(G(Fx,Fy,Fz)) \quad \text{for all } x,y,z \in X.$$

So in particular condition (2) holds.

Finally all the hypothesis of theorem are satisfied and $u = \frac{2}{3}$ is unique common fixed point of T and F.

Theorem (3.2): Let (X, G) be a metric space, suppose mapping $T, F: X \to X$, are G-R weakly commuting of type G_F and satisfy the following condition.

(2) T and F satisfy the (CLR$_F$) property.

(1) There exists nonnegative real constants $\alpha\ and\ \beta$ with $0 \leq \alpha + 2\beta < 1$ such that $x, y, z \in X$.

$$G\big(T(x), T(y), T(z)\big) \leq \alpha\, G\big(F(x), F(y), F(z)\big) + \beta \begin{Bmatrix} G\big(F(y), T(y), T(y)\big) + \\ G\big(F(z), T(z), T(z)\big) + \\ G\big(F(x), T(x), T(x)\big) \end{Bmatrix},$$

then T and F have a unique common fixed point.

Proof: The mapping T and F satisfy the (CLR_F) property, there exists in X a sequence

$$\lim_{n\to\infty} T\, x_n = \lim_{n\to\infty} F\, x_n = F(u), \quad \text{for some } u \in X.$$

First we will show that $Tu = Fu$. Suppose that $Tu \neq Fu$ then $d(Tu, Fu) > 0$. Using condition (2) with $x = u, y = u$ and $z = x_n$, we get

$$G\big(T(u), T(u), T(x_n)\big) \leq \alpha\, G\big(F(u), F(u), F(x_n)\big) + \beta \begin{Bmatrix} G\big(F(u), T(u), T(u)\big) + \\ G\big(F(x_n), T(x_n), T(x_n)\big) + \\ G\big(F(u), T(u), T(u)\big) \end{Bmatrix}.$$

Taking the limit as $n \to \infty$

$$G\big(T(u), T(u), F(u)\big) \leq (2\,\beta)\, G\big(F(u), T(u), T(u)\big),$$

which is true unless $G\big(T(u), T(u), F(u)\big) = 0$. That is $Tu = Fu$.

Since T and F are G-R weakly commuting of type G_F, then there exists a real constant R such that

$$G\big(TF(u), FT(u), TT(u)\big) \leq R\, G\big(T(u), F(u), T(u)\big) = 0.$$

Then $$TT(u) = TF(u) = FT(u) = FF(u).$$

Finally, we will show that $T(u)$ is common fixed point of T and F. We have

$$G\big(TT(u), T(u), T(u)\big) \leq \alpha\, G\big(FF(u), F(u), F(u)\big) + \beta \begin{Bmatrix} G\big(F(u), T(u), T(u)\big) + \\ G\big(F(u), T(u), T(u)\big) + \\ G\big(FT(u), TT(u), TT(u)\big) \end{Bmatrix}.$$

Since $FT(u) = TT(u)$ and $T(u) = F(u)$, then above inequality become

$$G\big(TT(u), T(u), T(u)\big) \leq \alpha\, G\big(TT(u), T(u), T(u)\big),$$

which holds unless $G\big(TT(u), T(u), T(u)\big) = 0$, so $TT(u) = FT(u) = T(u)$.

Then $T(u)$ is common fixed point.

Uniqueness of common fixed point is a direct consequence of condition (2).

Remark (3.1): Our result improves several known results including the results of Zead Mustafa (*Mustafa et al.,* 2012) for a pair of mapping in following ways:

(1) The completeness of space is not required,

(2) The completeness of subspace is not required even closedness of subspace is not required.

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