Original Article

On the Logical Origins of Quantum Mechanics Demonstrated By Using Clifford Algebra

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Abstract

Recently we have given proof of two theorems characterizing the Clifford algebra. By using such two theorems we have reformulated the well known von Neumann postulate on quantum measurements giving evidence of the algebraic manner in which quantum wave function collapse of quantum mechanics happens. In the present paper we introduce logic in Clifford algebra interpreting its idempotent as logical statements. Using the previously mentioned theorems we demonstrate that the two basic foundations of quantum mechanics, as the indeterminism and the quantum interference, do not arise from physics itself but from logic. We advance the principles that there are levels of our reality in which we lose our possibility of unconditionally define the truth. At this level of reality we cannot separate matter per se from the basic foundations of the logic that we use to describe it. This logical relativism does not characterize classical mechanics but quantum physics. According to Y.F. Orlov, at quantum level the truths of logical statements about dynamic variables become dynamic variables themselves.

Key Words: quantum cognition, quantum Clifford algebra, quantum mechanics of mental entities, quantum interference, quantum interference of logical statements, von Neumann theorem, interference of quantum logical statements.

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Introduction

Rather recently (Conte, 2010) we have given proof of two theorems on existing two Clifford algebras, the $A(S_i)$ that has isomorphism with that one of Pauli matrices, and the $N_{i,\pm1}$ where N_i stands for the dihedral Clifford algebra. The salient feature was that by using such two theorems, we showed that the $N_{i,\pm1}$ algebra may be obtained from the $A(S_i)$ algebra when we attribute a numerical value (+1 or −1) to one of the basic elements (e_1, e_2, e_3) of the $A(S_i)$. The arising physical model was that the $A(S_i)-$Clifford algebra refers to the representation of the general situation in quantum mechanics where the observer has no right to decide on the state of a two-state system while instead, through the operation represented by $N_{i,\pm1}$ algebra, he finally specifies which state is the one that will be or is being observed. The $A(S_i)-$algebra has as counterpart the description of quantum systems that in standard quantum mechanics are considered in absence of observation and quantum measurement while the $N_{i,\pm1}$ attend when a quantum measurement is performed on such system with advent of wave function collapse. There is another salient feature that needs to be outlined here. As said, under a Clifford algebraic profile, the quantum measurement with wave function collapse

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induces the passage in the considered quantum system from the $A(S_i)$ to $N_{i,+1}$ or to the $N_{i,-1}$ algebras: it is of interest from a mathematical and physical view points to observe that in the passage from $A(S_i)$ to $N_{1,\pm1}$, each $N_{1,\pm1}$ algebra has now its proper rules of commutation that are new and different respect to standard ones calculated in $A(S_i)$. Under the profile of a quantum measurement, wave function collapse is thus characterized, at least from an algebraic view point, just from such transition from standard to new commutation rules for the basic algebraic elements. This is an important feature that deserves careful physical consideration.

In (Conte, 2010) we re-examined also the well known von Neumann postulate on quantum measurement, and we gave a proper justification of such postulate by using such two theorems. In detail, we studied some application of the above mentioned theorems to some cases of interest in standard quantum mechanics, analyzing in particular a two state quantum system, the case of time dependent interaction of such system with a measuring apparatus and finally the case of a quantum system plus measuring apparatus developed at the order n=4 of the considered Clifford algebras and of the corresponding density matrix in standard quantum mechanics. In each of such cases, we found that the passage from the algebra $A(S_i)$ to $N_{i,\pm1}$ actually describes the collapse of the wave function. We concluded that the actual quantum measurement has as counterpart in the Clifford algebraic description, the passage from the $A(S_i)$ to the $N_{i,\pm1}$, reaching in this manner the objective to reformulate von Neumann postulate on quantum measurement and proposing at the same time a self-consistent formulation of quantum theory. The aim of the present paper is to propose a step on.

As it is well known, quantum mechanics runs about two basic foundations that are the indeterminism and the quantum interference. It is also well known that in 1932 John von Neumann (Von Neumann, 1932) gave proof that the projection operators and, in particular, quantum density matrices represent logical statements. We may say that he constructed matrix logic on the basis of quantum mechanics. In the present paper we will follow instead the statements and the results that were evidenced years ago by Y.F. Orlov (Orlov, 1978; 1982; 1994; 1996). By using the two previously mentioned theorems of Clifford algebra, we will attempt the inverse operation. We will construct the two basic foundations of quantum mechanics starting from logic and thus arriving to explain that quantum mechanics has logical origins. Not logic deriving from quantum mechanics as in von Neumann but quantum mechanics having logical origins. This is to say that the two basic foundations of quantum mechanics, the indeterminism and the quantum interference, may be explained on a purely logical basis.

The Clifford Algebra

Let us start with a proper definition of the 3-D space Clifford (geometric) algebra Cl_3. It is an associative algebra generated by three vectors e_1, e_2, and e_3 that satisfy the orthonormality relation

$$e_j e_k + e_k e_j = 2\delta_{jk} \qquad (1)$$

for $j,k,\lambda \in [1,2,3]$.

That is,

$e_\lambda^2 = 1$ and $e_j e_k = -e_k e_j$ for $j \neq k$

Let **a** and **b** be two vectors spanned by the three unit spatial vectors in $Cl_{3,0}$. By the orthonormality relation the product of these two vectors is given by the well known identity: $ab = a \cdot b + i(a \times b)$ where $i = e_1 e_2 e_3$ is a Clifford algebraic representation of the imaginary unity that commutes with vectors.

To give proof of $A(S_i)$, one must follow the approach that, starting with 1981, was developed by Y. Ilamed and N. Salingaros (Ilamed and Salingaros, 1981). All the details are given in ref. (Conte, 2010). Two basic postulates are required:

a. It exists the scalar square for each basic element:

$$e_1 e_1 = k_1,\ e_2 e_2 = k_2,\ e_3 e_3 = k_3 \text{ with } k_i \in \Re. \qquad (2)$$

In particular we have also the unit element, e_0, such that $e_0 e_0 = 1$.

b. The basic elements e_i are anticommuting elements, that is to say:

$$e_1e_2 = -e_2e_1 \;,\; e_2e_3 = -e_3e_2 \,,\; e_3e_1 = -e_1e_3 \,. \qquad (3)$$

with $e_ie_0 = e_0e_i = e_i$.

Following (Conte, 2010), we arrive to enunciate the following theorems.

Theorem n.1

Assuming the two postulates given in (a) and (b) with $k_i = 1$, the following commutation relations hold for such algebra :

$$e_1e_2 = -e_2e_1 = ie_3 \,;\, e_2e_3 = -e_3e_2 = ie_1 \,;$$

$$e_3e_1 = -e_1e_3 = ie_2 \;;\, i = e_1e_2e_3 \,,\, (e_1^2 = e_2^2 = e_3^2 = 1) \qquad (4)$$

They characterize the Clifford S_i algebra. We will call it the algebra $A(S_i)$.

Theorem n.2a

Assuming the postulates given in (a) and (b) with $k_1 = 1$, $k_2 = 1$, $k_3 = -1$, the following commutation rules hold for the new algebra obtained assigning to e_3 the numerical value +1:

$$e_1^2 = e_2^2 = 1 \,;\; i^2 = -1 \,;$$

$$e_1e_2 = i \,,\, e_2e_1 = -i \,,\, e_2i = -e_1 \,,\, ie_2 = e_1 \,,\, e_1i = e_2 \,,$$

$$ie_1 = -e_2 \qquad (5)$$

They characterize the Clifford N_i algebra. We will call it the algebra $N_{i,+1}$.

Theorem n.2b

Assuming the postulates given in (a) and (b) with $k_1 = 1$, $k_2 = 1$, $k_3 = -1$, the following commutation rules hold for new algebra obtained assigning to e_3 the numerical value −1 :

$$e_1^2 = e_2^2 = 1 \,;\; i^2 = -1 \,;$$

$$e_1e_2 = -i \;,\; e_2e_1 = i \,,\; e_2i = e_1 \,, ie_2 = -e_1 \,, e_1i = -e_2 \,,$$

$$ie_1 = e_2 \qquad (6)$$

They characterize the Clifford N_i algebra. We will call it the algebra $N_{i,-1}$

Let us consider now that the algebra $A(S_i)$ admits idempotent.

Consider two of such idempotent:

$$\psi_1 = \frac{1+e_3}{2} \text{ and } \psi_2 = \frac{1-e_3}{2} \qquad (7)$$

It is easy to verify that $\psi_1^2 = \psi_1$ and $\psi_2^2 = \psi_2$.

Let us examine the following algebraic relations:

$$e_3\psi_1 = \psi_1e_3 = \psi_1 \qquad (8)$$

$$e_3\psi_2 = \psi_2e_3 = -\psi_2 \qquad (9)$$

Similar relations hold in the case of e_1 or e_2 . From a conceptual point of view, looking at the (8) and (9) we reach only a conclusion. With reference to the idempotent ψ_1, the algebra $A(S_i)$, attributes to e_3 the numerical value of $+1$ while, with reference to the idempotent ψ_2, the algebra $A(S_i)$ attributes to e_3, the numerical value of -1.

However, considering the attribution $e_3 \to$ +1, we have that $A(S_i)$ links itself to the $N_{1,+1}$ algebra that now holds, with new commutation relations given in (5) and three new basic elements that are (e_1,e_2,i) instead of (e_1,e_2,e_3). Considering instead the attribution $e_3 \to$ -1, we have that $A(S_i)$ links itself to the $N_{1,-1}$ algebra that now holds, with new commutation relations given in (6) and three new basic elements that are (e_1,e_2,i) instead of (e_1,e_2,e_3). Let us examine now another feature.

A generic member of algebra $A(S_i)$ is given by

$$x = \sum_{i=0}^{4} x_ie_i \qquad (10)$$

with x_i pertaining to some field $\Re$ or C .

We may transform Clifford members of $A(S_i)$ using linear homogeneous transformations so that

$$x' = SxS^+ \qquad (11)$$

Generally speaking we take

$$S(a,b) = \frac{a+a^*}{2} + (\frac{b+b^*}{2})e_1 + (\frac{i(b-b^*)}{2})e_2 + (\frac{a-a^*}{2})e_3 \qquad (12)$$

$$S^+(a,b) = \frac{a+a^*}{2} + (\frac{b^*-b}{2})e_1 - (\frac{i(b+b^*)}{2})e_2 + (\frac{a^*-a}{2})e_3 \qquad (13)$$

that are members of the $A(S_i)$ algebra. Taking the relative matrix representation, one acknowledges easily that we are considering the SU_2 group.

In analysis of Clifford algebraic $A(S_i)$ transformed members we usually use (Conte, 2001) the following transformation

$$S(a,b) = a + b e_i \,;\, S^{+}(a,b) = a^{*} + b^{*} e_i \,;\, i = 1,2,3 \quad (14)$$

with

$$aa^{*} + bb^{*} = 1 \quad \text{and} \quad ab^{*} + a^{*}b = 0 \quad (15)$$

and

$$SS^{+} = S^{+}S = 1$$

Applying the (14) with $i = 3$ we obtain the three new Clifford basic elements:

$$\bar{e}_1 = S e_1 S^{+} \;;\; \bar{e}_2 = S e_2 S^{+} \;;\; \bar{e}_3 = S e_3 S^{+} \quad (16)$$

that in matrix notation assume the following forms

$$\bar{e}_1 = \begin{pmatrix} 0 & \omega \\ \vartheta & 0 \end{pmatrix}, \bar{e}_2 = \begin{pmatrix} 0 & -i\omega \\ i\vartheta & 0 \end{pmatrix}, \bar{e}_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = e_3 \quad (17)$$

with

$$aa^{*} - bb^{*} - 2a^{*}b = \vartheta \quad \text{and}$$
$$aa^{*} - bb^{*} + 2a^{*}b = \omega \quad (18)$$

Obviously, the (16) or the (17) represent a new basic triad of Clifford algebraic elements in $A(S_i)$ with rules given in the (4). In detail, we may apply to the (16) the theorems n.1, n. 2a, and n. 2b.

These are the basic notations that we will use in the present paper. However, before of concluding the present section, we would add still some final elaboration that of course was discussed in detail by us previously in (Conte, 2010), and that we retain necessary to take again here in consideration, briefly, because it touches directly the argument that is at the basis of the present paper. Consider again the algebra $A(S_i)$ as given in (4).

If we consider the $e_i\,(i = 1,2,3)$ as abstract entities, we may conclude that they have an intrinsic randomness that represents their essential irreducible nature. Their being $e_i^2 = 1$, implies substantially that they have the intrinsic and indeterminate potentiality of being (± 1) and to actually assume or the numerical value +1 or the numerical value –1. As repeatedly outlined previously, the theorems n.2a and n.2b regulate in substance such transition from potentiality to actualization. Since the e_i are abstract entities, having the potentiality that we may attribute them the numerical values, $+1$ or -1, they have such intrinsic and irreducible randomness that we may characterize admitting to be $p_1(+1)$ the probability that e_1 assumes the value $(+1)$ and $p_1(-1)$ the probability that it assumes the value -1. In this manner we may write a mean value that is given by

$$< e_1 > = (+1)p_1(+1) + (-1)p_1(-1) \quad (19)$$

Considering the same corresponding notation for the two remaining basic elements, we may introduce the following mean values:

$$< e_2 > = (+1)p_2(+1) + (-1)p_2(-1)\,, \quad (20)$$
$$< e_3 > = (+1)p_3(+1) + (-1)p_3(-1).$$

We may write that

$$-1 \le < e_i > \le +1 \quad i = (1,2,3) \quad (21)$$

Selected the following generic element of the algebra $A(S_i)$:

$$x = \sum_{i=1}^{3} x_i e_i \qquad x_i \in \Re \quad (22)$$

note that

$$x^2 = x_1^2 + x_2^2 + x_3^2 \quad (23)$$

Its mean value results to be

$$< x > = x_1 < e_1 > + x_2 < e_2 > + x_3 < e_3 > \quad (24)$$

Let us call

$$a = (x_1^2 + x_2^2 + x_3^2)^{1/2} \quad (25)$$

so that we may attribute to x the value $+a$ or $-a$

We have that

$$-a \le x_1 < e_1 > + x_2 < e_2 > + x_3 < e_3 > \le a \quad (26)$$

The (26) must hold for any real number x_i, and, in particular, for

$$x_i = < e_i > \quad (27)$$

so that we have that

$$x_1^2 + x_2^2 + x_3^2 \le a \quad (28)$$

that is to say

$$a^2 \le a \quad \to a \le 1 \quad (29)$$

so that we have the fundamental relation

$$< e_1 >^2 + < e_2 >^2 + < e_3 >^2 \leq 1 \quad (30)$$

This is the basic relation we are writing in our Clifford algebraic $A(S_i)$ algebraic quantum scheme.

Let us observe some important things:

(a) First of all it links the Clifford algebra $A(S_i)$ with the $N_{1,\pm 1}$. In absence of a direct attribution of a numerical value to one basic algebraic element e_i, the (30) holds.

(b) If we attribute instead a definite numerical value to one of the three basic entities, as example we attribute to e_3 the numerical value +1, we have $< e_3 >= 1$, the (30) operates now in the $N_{i,+1}$ algebra, and is reduced to

$$< e_1 >^2 + < e_2 >^2 = 0 \;,\; < e_1 >=< e_2 >= 0 \;, \quad (31)$$

and we have complete, irreducible, indetermination for e_1 and for e_2.

(c) Finally, the (30) affirms that we never can attribute simultaneously definite numerical values to two basic non commutative elements e_i. Still let us examine another important consequence. As previously evidenced, in Clifford algebra $A(S_i)$ we have idempotents. Let us consider again two of such idempotents:

$$\psi_1 = \frac{1+e_3}{2} \quad \text{and} \quad \psi_2 = \frac{1-e_3}{2} \quad (32)$$

Let us consider the mean values of (32). We have that

$$2 < \psi_1 >= 1 + < e_3 > \text{ and}$$
$$2 < \psi_2 >= 1 - < e_3 > \quad (33)$$

Using the (20) we obtain that

$$p_3(+1) = \frac{1 + < e_3 >}{2} \quad \text{and} \quad p_3(-1) = \frac{1 - < e_3 >}{2} \quad (34)$$

Therefore, we have that

$$p_3(+1) =< \psi_1 > \text{ and } p_3(-1) =< \psi_2 > \quad (35)$$

This is to say that probabilities $p_3(+1,-1)$ are the mean values of the idempotents. The same result holds obviously when considering the basic elements e_1 or e_2. Considering that in quantum mechanics (Born probability rule), given wave functions $\varphi_{+,-}$, we have

$$\left|\varphi_{+,-}\right|^2 = p_{+,-} \quad (36)$$

we conclude that

$$\varphi_3(+) = \sqrt{< \psi_1 >} e^{i\vartheta_1} \text{ and } \varphi_3(-) = \sqrt{< \psi_2 >} e^{i\vartheta_2} \quad (37)$$

These are very surprising results. In brief we have delineated a scheme of quantum mechanics without invoking ever also only one physical principle or physical basic result that of course signed the advent of quantum mechanics. Certainly we have given only a rough quantum mechanical scheme, we have obtained something that it is similar to what T.J Jordan properly called time ago a bare bone skeleton of quantum mechanics when he realized in a textbook an exposition of quantum mechanics in simple matrix form (Jordan, 1985). However, it remains very surprising the realization of such scheme without invoking also only one physical foundation of quantum theory. Note that we would also have to continue in the elaboration of such scheme introducing Heisenberg like indetermination principle, Schrödinger-like equation, and other essential features of such theory (Conte, 2001). In substance, such results recall preliminarily the question that we outlined in the introduction of the present paper, and relating the problem about what are the actual origins of quantum mechanics.

The Logical Origins of Indeterminism and Quantum Interference in quantum mechanics

We start the present section recalling that Y. O. Orlov was the first to introduce the so called Wave Logic based on quantum mechanics (Orlov, 1978; 1982; 1994; 1996) starting with his celebrated paper that he wrote from his prison camp 37-2 Urals, USSR and published just on such journal. We will not re-discuss here all the features of such logic. The reader is sent back to the original papers signed in (Orlov, 1978; 1982; 1994; 1996) to delineate the basic rules of such logic. We do not report them here for brevity. We may consider some fundamental steps in our discussion.

The first is the original statement introduced by von Neumann in 1932 (Von

Neumann, 1932) who showed that the projection operators, say Λ, having the property of idempotence, $\Lambda(\Lambda-1)=0$, are logical statements.

The second is that we identified such projection operators, here intended as members of the Clifford $A(S_i)$ algebra, and the important examples of such idempotents were given by us in the (7). Finally, Orlov (1978; 1982; 1994; 1996) introduced in his logic, the first, elementary (atomic), statement λ_k ($k=1,2,...N$) representing it by 2×2 diagonal matrices

$$\lambda_k=\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix},\quad \lambda_k^2=\lambda_k \tag{38}$$

λ_k is a statement that may be true (eigenvalue +1) or false (eigenvalue 0) and the negation of λ_k is $\bar{\lambda}_k$ represented by the matrix

$$\bar{\lambda}_k=1-\lambda_k=\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix};\quad \bar{\lambda}_k^2=\lambda_k \tag{39}$$

The other basic features of such logic were discussed in detail by Orlov in (Orlov, 1978; 1982; 1994; 1996). It is important to evidence here that in such new Clifford scheme, λ_k and $\bar{\lambda}_k$ (that are the idempotents introduced by us in the (7) and pertaining thus to the algebra $A(S_i)$ and subjected to the two theorems n.1 and n.2a and n.2b), represent for us logical statements in accord with von Neumann and Orlov. Let us sketch briefly the scheme of such logic in our paper.

Let us consider such idempotents given in the $A(S_i)$ algebra. In (Conte, 1993) we showed the basic theorem for such idempotents. Given the member ρ of $A(S_i)$: $\rho=a+be_1+ce_2+de_3$ the necessary and sufficient condition for having $\rho^2=\rho$ are that $a=1/2$ and $b^2+c^2+d^2=1/4$. (40)

By using such necessary and sufficient conditions we have examples of idempotents:

$$\lambda_0=\frac{1+e_3}{2} \tag{41}$$

that in our logic scheme by Clifford $A(S_i)$ algebra, represents the elementary logic statement, discussed by von Neumann and by Orlov. In our algebraic scheme: when it has the potential meaning to be true or false, we have the (41) in $A(S_i)$. By application of the theorem n2a and /or n.2b we pass instead in Clifford algebra $N_{1,\pm1}$ and λ_0 becomes or true or false (respectively assumes directly the numerical value or +1 or 0).

By Application of the theorem given in (Conte, 1993), other idempotents, and thus other logical statements, are given by supporting the following conditions

$a=1/2$, $b=0$, $c=\pm\frac{sen\gamma}{2}$, $d=\pm\frac{\cos\gamma}{2}$

or

$a=1/2$, $b=\pm\frac{sen\gamma}{2}$, $c=0$, $d=\pm\frac{\cos\gamma}{2}$ (42)

or still

$a=1/2$, $b=\pm\frac{sen\gamma}{2}$, $c=\pm\frac{\cos\gamma}{2}$, $d=0$

and

$a=1/2$, $b=c=d=\frac{1}{2\sqrt{3}}$.

Looking at the (42), we deduce that, generally speaking, we may construct several, different and more complex statements $\lambda_0,\lambda_1,\lambda_2,................,\lambda_n$ that may result to be true or false or in their potential state of logical indetermination until a numerical value is attributed directly to them by the theorem 2. By subsequent application of the (11), we will have

$$\lambda_0\to\lambda_j\to\cdots\cdots\lambda_k \tag{43}$$

and for each pair of logical statements, we may calculate $\lambda_j\lambda_0$ that will be still an element of the $A(S_i)$ Clifford algebra. Applying to such member the theorem n.2a or n.2b in $N_{i,\pm1}$, we will calculate the probability that the logical statement λ_j may be predicted to be true or false starting truth value of λ_0. Call the logical statement λ_0 by A, and we mean that it is true when we write $A=+1$. Otherwise, we intend that it is false, when we write $A=-1$. Call the logical statement λ_j by B, being it true when we write $B=+1$, and false when $B=-1$. By the procedure of application of theorem n.2a, n.2b to $\lambda_j\lambda_0$, and thus passing from $A(S_i)$ to $N_{i,\pm1}$, assuming as example A to be true, we will calculate the probability that the logical statement B is predicted to be true being A true, or the probability that the logical statement B is predicted to be false being A true. This is to say that by this procedure we evaluate the probabilities $p(B=+1/A=+1)$ or $p(B=-1/A=+1)$.

Note two important features that we are delineating by our Clifford logic scheme. Let us admit that we estimate $\lambda_1\lambda_0$ as member of $A(S_i)$. Passing from $A(S_i)$ to $N_{i,\pm1}$, two possible cases will result possible. Or we will obtain a new element in $N_{i,\pm1}$ that will be reduced directly to a numerical value, and consequently we will conclude to have directly estimated the probability $p(B/A)$ or it will result instead that we obtain an element of $N_{i,\pm1}$ that of course does not assume directly a numerical value. This is a case in which we will conclude for what we will call the *incompetence* of the logical statement λ_0 to predict probability for true or false value of λ_1. As we will see in detail in the subsequent calculations, we will estimate Clifford members as $\lambda_1\lambda_0$, $\lambda_2\lambda_1$, computing each time the classical probabilities $p(B/A)$, $p(C/B)$ and obviously we will expect at the end to find the same calculated value of probability, $p(C/A)$, corresponding to $\lambda_2\lambda_0$. In performing calculations corresponding to such classical scheme of probability, each time (that is to say in $\lambda_1\lambda_0$, and $\lambda_2\lambda_1$), we will force the subordinate logical statement to result or true or false. Following this procedure, we will exclude a priori the important cases in which the *incompetence* for the subordinate statement to be predicted might appear. Computing instead directly $\lambda_2\lambda_0$, passing through the intermediate passage, represented by the intermediate logical statement λ_1, it will remain free the possibility for the statement λ_0 to have *incompetence* in predicting probability for λ_1 so that the two arising logic schemes, one pertaining to classical logic and to classical probability and the other pertaining instead to an extended logical scheme in which the possibility for *incompetence* remains, could result possibly profoundly different. We retain that here it should be the root of the profound difference existing between classical and quantum framework. The first one eliminates any possible source of indeterminism that instead represents an essential component in logic and semantic that consequently cannot be ignored. We may now pass to the calculations. Let us start considering the following basic statement

$$\lambda_0 = \frac{1+e_3}{2} \tag{44}$$

and let us calculate the new logical statement λ_1

$$\lambda_1 = S\,\lambda_0 S^+ \tag{45}$$

with S and S^+, Clifford algebraic elements in $A(S_i)$, given in the following manner

$$S = \cos(\beta_1/2) + i\,sen(\beta_1/2)e_2 \text{ and}$$

$$S^+ = \cos(\beta_1/2) - i\,sen(\beta_1/2)e_2\,; \quad S\,S^+ = 1\,. \tag{46}$$

We obtain that

$$\lambda_1 = \frac{1}{2} - \frac{1}{2}sen\beta_1 e_1 + \frac{1}{2}\cos\beta_1\, e_3 \tag{47}$$

Note that it is one of the idempotents previously identified in (42). It is a logical statement.

Obviously it is $\lambda_1^2 = \lambda_1$. Let us observe also that we have found three new basic elements

$$\bar{e}_1 = \cos\beta_1\, e_1 + sen\beta_1\, e_3\,, \bar{e}_2 = e_2\,,$$

$$\bar{e}_3 = -sen\beta_1\, e_1 + \cos\beta_1\, e_3 \tag{48}$$

These are three new basic elements that pertain to the $A(S_i)$ algebra, obeying the same rules given previously in (4). Note of course that the logical statement λ_1 may be rewritten as

$$\lambda_1 = \frac{1+\bar{e}_3}{2} \tag{49}$$

In such new scheme with three basic elements $(\bar{e}_1, \bar{e}_2, \bar{e}_3)$, λ_1 may result to be or true or false by applying to it the $N_{i,\pm1}$. Now, obeying to the same principle, let us calculate

$$\lambda_2 = S\,\lambda_1\,S^+ \tag{50}$$

where this time we have
$S = \cos(\beta_2/2) + i\,sen(\beta_2/2)e_2$ and

$$S^+ = \cos(\beta_2/2) - i\,sen(\beta_2/2)e_2;\ S\,S^+ = 1 \tag{51}$$

We obtain that

$$\lambda_2 = \frac{1}{2} - \frac{1}{2}sen(\beta_1+\beta_2) + \frac{1}{2}\cos(\beta_1+\beta_2)e_3 \tag{52}$$

Again it is that it is one of the idempotents given in (40)

$$\lambda_2^2 = \lambda_2 \tag{53}$$

According to our approach it is a new logical statement. Here we have found still three new basic elements in $A(S_i)$ algebra

$$\hat{e}_1 = \cos(\beta_1+\beta_2)e_1 + sen(\beta_1+\beta_2)e_3;\ \hat{e}_2 = e_2;$$
$$\hat{e}_3 = -sen(\beta_1+\beta_2)e_1 + \cos(\beta_1+\beta_2)e_3 \tag{54}$$

These are still three new basic elements of the $A(S_i)$ algebra, obeying the same rules given previously in (4). Again we may write the new logical statement

$$\lambda_2 = \frac{1+\hat{e}_3}{2} \tag{55}$$

that may result true or false by using $N_{1,\pm1}$.

We have now calculated the three Clifford elements $\lambda_0, \lambda_1, \lambda_2$. For easiness, let use the previous scheme, and let us indicate such logic statements by A, B, C. We agree to write $A = 1$ when the logical statement A is true, and $A = -1$ when it is false. We adopt the same convention for the logical statements B and C, respectively.

Considering the (47) and the (44) and the (4), let us calculate the Clifford member $\lambda_1\lambda_0$. After calculations we obtain that

$$\lambda_1\lambda_0 = (\frac{1}{4} + \frac{1}{4}\cos\beta_1) + \frac{1}{4}sen\beta_1(-e_1 + ie_2) + (\frac{1}{4} + \frac{1}{4}\cos\beta_1) \tag{56}$$

Let us observe again that this is a member of $A(S_i)$. Let us admit now that we intend to calculate the probability that λ_1 is true when λ_0 is true. This is to say $p(B = +1/A = +1)$. On the basis of the two theorems shown in (Conte, 2010), reassumed previously, we have that the algebra $A(S_i)$ no more holds, and instead the algebra $N_{1,+1}$ holds with rules given in (5) and here repeated:

$$e_1^2 = e_2^2 = 1;\ i^2 = -1;$$
$$e_1e_2 = i,$$
$$e_2e_1 = -i,$$
$$e_2 i = -e_1, ie_2 = e_1, e_1 i = e_2,\ ie_1 = -e_2 \tag{57}$$

Note that by applying the (57) in the (56) we have not problems in attributing a direct numerical value to (56) owing to the presence of the term

$$\frac{1}{4}sen\beta_1(-e_1 + ie_2)$$

that directly goes to zero. So we finally obtain

$$\lambda_1\lambda_0 = \frac{1}{2} + \frac{1}{2}\cos\beta_1 = \cos^2(\beta_1/2) \tag{58}$$

In conclusion we obtain that

$$p(B = +1/A = +1) = \cos^2(\beta_1/2) \tag{59}$$

Note that we have not possible *incompetence* in this case. Given the logical statement λ_0 we always may estimate the probability of the logical statement λ_1 to be true or false being λ_0 true or false. Repeating the same procedure we obtain that

$$p(B = -1/A = +1) = sen^2(\beta_1/2) \tag{60}$$

with

$$p(B = +1/A = +1) + p(B = -1/A = +1) = 1 \tag{61}$$

We have not indetermination (that is to say *incompetence*) in this case. Considering now the (47) and the (52) with the algebraic rules given in (4), let us calculate the Clifford algebraic element $\lambda_2\lambda_1$. We obtain that

$$\lambda_2\lambda_1 = (\frac{1}{4} + \frac{1}{4}sen\beta_1 sen(\beta_1+\beta_2) + \frac{1}{4}\cos\beta_1\cos(\beta_1+\beta_2)) + (-\frac{1}{4}sen\beta_1 - \frac{1}{4}sen(\beta_1+\beta_2))e_1 + \frac{ie_2}{4}\cos\beta_1 sen(\beta_1+\beta_2) + \frac{ie_2}{4}(-sen\beta_1\cos(\beta_1+\beta_2)) + \frac{1}{4}\cos\beta_1\cos(\beta_1+\beta_2)e_3 \tag{62}$$

This is still a member of $A(S_i)$ Clifford algebra. Consider now that we intend to calculate the probability that λ_2 is true when λ_1 is true. This is to say $p(C = +1/B = +1)$. By application of $N_{i,\pm1}$, we obtain for the following expression

$$\lambda_2\lambda_1 = (\frac{1}{4}+\frac{1}{4}sen\beta_1 sen(\beta_1+\beta_2)+\frac{1}{4}\cos\beta_1\cos(\beta_1+\beta_2)) \quad (63)$$
$$+(-\frac{1}{4}sen\beta_1 - \frac{1}{4}sen(\beta_1+\beta_2))e_1+\frac{ie_2}{4}\cos\beta_1 sen(\beta_1+\beta_2)$$
$$+\frac{ie_2}{4}(-sen\beta_1\cos(\beta_1+\beta_2))+\frac{1}{4}\cos\beta_1\cos(\beta_1+\beta_2)$$

As it is immediately verified, it no more gives a direct numeric expression since. In fact, by application of the rules given in (5), the terms containing this time e_1 and e_2 no more disappear. This is a case of *incompetence* for λ_1 to estimate probability for λ_2. The only way remaining to obtain a direct numerical value it is that we consider

$$sen\beta_1 = 0 \quad (64)$$

and in this case we have that

$$\lambda_2\lambda_1 = (\frac{1}{4}+\frac{1}{4}\cos\beta_2)+\frac{1}{4}sen\beta_2(-e_1+ie_2) \quad (65)$$
$$+(\frac{1}{4}+\frac{1}{4}\cos\beta_2)e_3$$

It is still a member of $A(S_i)$. In order to calculate $p(C=+1/B=+1)$, we remember the $A(S_i)$ algebra no more holds, and instead we must use the $N_{i,+1}$ with rules given in the (5). Consequently, we obtain

$$\lambda_2\lambda_1 = \frac{1}{2}+\frac{1}{2}\cos\beta_2 \quad (66)$$

that is to say that

$$p(C=+1/B=+1) = \cos^2(\beta_2/2) \quad (67)$$

and the probability is now estimated.

Using the same procedure for calculations, we may obtain the probability that λ_2 is true if λ_1 is false. We obtain

$$p(C=+1/B=-1) = sen^2(\beta_2/2) \quad (68)$$

In conclusion we have:

$$p(B=+1/A=+1)\,p(C=+1/B=+1)$$
$$+\,p(B=-1/A=+1)\,p(C=+1/B=-1) = \quad (69)$$
$$\cos^2(\beta_1/2)\cos^2(\beta_2/2)+sen^2(\beta_1/2)sen^2(\beta_2/2)$$

It remains now to calculate $p(C=+1/A=+1)$ that is to say the probability that the logical statement λ_2 is true if λ_0 is true. Let us calculate $\lambda_2\lambda_0$ using the (44), and the (52) with rules given in (4). We obtain that

$$\lambda_2\lambda_0 = (\frac{1}{4}+\frac{1}{4}\cos(\beta_1+\beta_2))$$
$$+\frac{1}{4}sen(\beta_1+\beta_2)(-e_1+ie_2) \quad (70)$$
$$+(\frac{1}{4}+\frac{1}{4}\cos(\beta_1+\beta_2))\ e_3$$

This is still a member of $A(S_i)$. In order to calculate $p(C=+1/A=+1)$, the $A(S_i)$ algebra no more holds, and we have to pass to $N_{i,+1}$ with the rules given in (5). We obtain

$$\lambda_2\lambda_0 = \frac{1}{2}+\frac{1}{2}\cos(\beta_1+\beta_2) \quad (71)$$

that is to say that

$$p(C=+1/A=+1) = \cos^2((\beta_1+\beta_2)/2) \quad (72)$$

Note that also in this case we have not had problems. The application of $N_{i,\pm1}$ has enabled us to attribute a direct numerical value to $\lambda_2\lambda_0$ and to $p(C=+1/A=+1)$.

However, in calculating $\lambda_2\lambda_0$, we have transformed λ_0 in λ_1 and λ_1 in λ_2. Executing such transformations we did not impose the restrictions, given in (64), that instead we introduced in (63) in order to eliminate *incompetence* or, that is to say, indeterminism. On this basis we calculated the (67).

We are now in the condition to summarize our results

I. According to von Neumann who stated that projection operators and, in particular, quantum density matrices, represent logical statements, transferring such argument at an algebraic Clifford level, we have assumed that the idempotents like the (41) and the (42), given in the algebra $A(S_i)$, represent logical statements with this algebra characterized by the basic rules given in the (4) by the theorem n.1

II. We have also assumed that e_3, the third basic element of (e_1, e_2, e_3) of the given $A(S_i)$ algebra, represents an atomic proposition of classical logic as well as Orlov assumed in 1982. It is represented by the third component of the Pauli algebra.

III. According to our previous paper (Conte, 2010) we have also considered that in order to evaluate the truth or false value of a given logical statement, we have to pass from the algebra $A(S_i)$ to the algebra $N_{1,\pm 1}$ whose existence is shown by the theorems n.2a and n.2b. It has new rules given respectively in (5) and in (6).

IV. Still according to Orlov, we have rejected the hypothesis that it exists a definition of absolute logical truth. Instead we have admitted a principle of relativity of logical truth values, assuming that such principle is realized by using linear homogeneous transformations of Clifford algebraic $A(S_i)$ elements. We have described in detail such basic feature starting by the (11).

V. There is still a statement that follows from using theorems n.1 and n2a and n.2b in our logical approach by Clifford algebra. It is that there is not exist in our reality the possibility of always defining unconditionally a truth or its relative and subordinate probability. Let us redefine better the concept of *incompetence* that we have previously delineated. Let the two statements A and B be represented by λ_i and λ_j, respectively. As said, by applying $N_{i,\pm 1}$ to $\lambda_j\lambda_i$ we may calculate $p(B/A)$. According to standard logic and reasoning, we have only two possibilities . One is the case in which we calculate the probability that being A true, also B is true or being A false also B is false. The second case is that being A true, B is false or being A false, B is true. To such before mentioned possibilities we must add also the case in which A has not the *competence* to establish the logical truth values of B. We have in this case a situation of intrinsic indetermination.

On the basis of such assumptions, we have calculated $\lambda_1\lambda_0$ and $\lambda_2\lambda_1$, and to such members of $A(S_i)$ algebra we have applied the theorem n.2a and n.2b obtaining the probabilities

$$p(B=+1/A=+1)=\cos^2(\beta_1/2);$$
$$p(B=-1/A=+1)=sen^2(\beta_1/2)$$

and still

$$p(C=+1/B=+1)=\cos^2(\beta_2/2);$$
$$p(C=+1/B=-1)=sen^2(\beta_2/2) \quad (73)$$

Using the same procedure of calculation and elaboration we calculated also $\lambda_2\lambda_0$ and the probability p(C=+1/A=+1) expecting to find as in classical probability theory that

$$p(C=+1/A=+1)=\cos^2(\beta_1/2)\cos^2(\beta_2/2)+sen^2(\beta_1/2)sen^2(\beta_2/2) \quad (74)$$

Instead, according to the (72), we found a non classical probability result that is

$$\hat{p}(C=+1/A=+1)=\cos^2((\beta_1+\beta_2)/2) \quad (75)$$

In conclusion we had that

$$p(C=+1/A=+1)\neq\hat{p}(C=+1/A=+1) \quad (76)$$

and

$$\hat{p}_{non.classical}(C=+1/A=+1)=p_{classical}(C=+1/A=+1)-\frac{1}{2}sen\beta_1 sen\beta_2 \quad (77)$$

The (77) is also in accord with the results that were obtained previously by Orlov (1978; 1982; 1994; 1996).

In conclusion we find the presence of the interference term

$$-\frac{1}{2}sen\beta_1 sen\beta_2 \quad (78)$$

In this manner we have reached an interesting conclusion. We have given proof of quantum interference and of indeterminism by using only logic realized in the framework of Clifford algebra and adopting only the two previously mentioned theorems n.1 and n.2a and n.2b.

It is well known that quantum mechanics runs about two basic foundations that are just the indeterminism and the quantum interference. We have obtained both such two foundations without adopting physics, and, in detail, without adopting neither one of the quantum physic principles or rules that characterize the physical basis of quantum theory. Therefore, the origins of indeterminism and of quantum interference are not in physics itself but in the logic introduced in our Clifford logic scheme.

There is still another important feature that we have to outline here. To this purpose, let us perform now the last calculation. Look again to the (75) that we re-write here:

$$\hat{p}(C=+1/A=+1)=\cos^2((\beta_1+\beta_2)/2) \quad (79)$$

From the (79) it follows that

$$\sqrt{\hat{p}(C=+1/A=+1)}=\cos((\beta_1+\beta_2)/2)) \\ =\cos(\beta_1/2)\cos(\beta_2/2)-sen(\beta_1/2)\,sen(\beta_2/2) \quad (80)$$

that on the basis of the (73) may be re-written in the following manner:

$$\sqrt{\hat{p}(C=+1/A=+1} = \\ \sqrt{p(B=+1/A=+1}\sqrt{p(C=+1/B=+1}+ \\ \sqrt{p(B=-1/A=+1)}\sqrt{p(C=+1/B=-1)} \quad (81)$$

that is the rule of probability amplitudes in quantum mechanics.

Conclusions

In this paper we have only used Clifford algebra and two theorems that we recently showed (Conte, 2010). By it we first realized a rough scheme of quantum mechanics; see the (17)-(35), that of course we discussed in detail in (Conte, 2010). Rather than using the expression a rough scheme of quantum mechanics, we may say that by it we constructed a bare bone skeleton of quantum mechanics using a phrase that time ago T.F Jordan adopted in his textbook realizing quantum mechanics in a simple matrix form.

Using Clifford algebra we introduced a Clifford representation of mathematical logic using the idempotents of such algebra as logic statements. For brevity we did not itemize all the features of logic propositions because they are at last well known and in particular they were listened by Orlov (Orlov, 1978; 1982; 1994; 1996) when time ago he introduced his Wave Calculus based on Wave logic. If interested, the reader may goes back reading in detail such fundamental papers.

By using such Clifford formulation we reached substantially there results:

▪We showed that the basic foundations of quantum mechanics, that are the indeterminism and the quantum interference, may be deduced on the basis of a purely logical basis constructed by Clifford algebra. In detail, in deducing that the origins of the most fundamental quantum phenomena as the indeterminism and quantum interference, lies in the logic, we did not invoke any physical element, not one physical foundation of quantum mechanics as quantum action, wave functions, observables, and so on. We did not use traditional quantum mechanical physics but only logic principles. So, we may conclude that the origins of some basic fundamental quantum phenomena are not in physics itself but in the logic as it may run by using Clifford algebra.

▪The other interesting result that we reached is that we showed the rule of probability amplitudes, as given in (79)-(81), without recovering quantum physics but our Clifford Logic scheme. As it is well known, also the rule of square probability amplitudes represents another basic foundation of quantum mechanics. Also this result is in accord with the elaboration that was previously obtained by Orlov (1978; 1982; 1994; 1996).

In conclusion, we may say that the basic features of quantum mechanics result to be consistent with the logic formulation that we introduced by using the Clifford algebra. In brief, J. von Neumann showed that projection operators of quantum mechanics can be interpreted as logical statements. We may say that he constructed a matrix logic derived from quantum mechanics. In some sense we reached an inverted objective. We showed that quantum mechanics is constructed on the basis of a logic realized by Clifford algebra. Consequently, it arises as necessary the problem to ask what the reasons are because such logical relativism arises in quantum mechanics while instead such relativism does not exist in classical physics. We have to explain the reason because logic, and thus language, semantic and human cognition, play such a fundamental role in quantum mechanics while only an auxiliary support may be found in classical physics. Still according in some manner with Orlov (Orlov, 1978; 1982; 1994; 1996), we may explain such feature considering that in quantum mechanics we no more can separate the logic from the features of "matter per se". In quantum mechanics the logic assumes the same importance as the features of what is being described. We are at levels of our reality in which the truths of logical statements about dynamic variables become dynamic variables themselves so that a profound link is established from its starting in this theory between physics and logic. Of course, Orlov gives an excellent example of this fundamental feature of quantum reality observing that in quantum mechanics we have from one hand physical observables linked to dynamic physical variables, call one by Q, as example. On the other hand we have the corresponding projectors Λ_Q that of course represent logical statements. The Λ_Q always commutes with $\hat{Q}$ that is the hermitean operator connected to Q.

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