

# Quantum Cybernetics and Complex Quantum Systems Science: A Quantum Connectionist Exploration

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## ABSTRACT

Quantum cybernetics and its connections to complex quantum systems science is addressed from the perspective of quantum artificial neural networks as complex quantum computing systems. In this way, the notion of an autonomous quantum computing system is introduced in regards to quantum artificial intelligence, and applied to quantum artificial neural networks, considered as autonomous quantum computing systems, which leads to a quantum connectionist framework within quantum cybernetics for complex quantum computing systems. Several examples of quantum feedforward neural networks are addressed in regards to Boolean functions' computation, multilayer quantum computation dynamics, entanglement and quantum complementarity. The examples provide a basis for a reflection on the role of quantum artificial neural networks as a general framework for addressing complex quantum systems that perform network-based quantum computation, possible consequences are drawn regarding quantum technologies, as well as fundamental research in complex quantum systems science, neuroscience and quantum biology.

**Key Words:** quantum cybernetics, quantum artificial neural networks, complex quantum systems, quantum connectionism

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## 1. Introduction

The current work is aimed at the expansion of connectionist-based quantum cybernetics and complex quantum systems science by addressing directly two central threads within quantum artificial intelligence research, namely: the notion of an autonomous quantum computing system

(AQCS) and a generalization to the quantum setting of the connectionist framework for artificial intelligence, addressing both the computational aspects of quantum artificial neural networks (QuANNs) as well as the implications for complex quantum systems science and for quantum technologies.

In section 2., a general background on quantum cybernetics is provided, including its connection with quantum computation and complex quantum systems science. In section 3., the notion of an autonomous quantum computing system (AQCS), in connection to quantum artificial intelligence, is introduced. In section 4., quantum artificial neural networks (QuANNs)

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theory is reviewed and QuANNs are addressed as AQCSs. In particular, quantum feedforward neural networks are worked upon so that major points such as the problem of computation of Boolean functions and neural network computational complexity, multiple layers and the issue of complementarity are addressed. In section 5., a reflection is presented, addressing the main results and the connection between complex quantum systems science, quantum connectionist approaches and quantum technologies.

## 2. Quantum Cybernetics and Quantum Computation

Cybernetics began as a post-World War II scientific movement gathering researchers from different disciplinary backgrounds (Lafontaine, 2004). Norbert Wiener defined cybernetics as the scientific study of control and communication in the animal and the machine (Wiener, 1948). Intersecting with general systems science, cybernetics grew as a transdisciplinary research field dealing with issues like feedback, control and communication, self-organization and autonomy in natural and artificial systems (Lafontaine, 2004). In artificial systems these issues are raised to the central research problem of an autonomous computing system endowed with an artificial intelligence that may allow it to select a program and adapt to different conditions.

Up until the end of the 20th Century, cybernetics was strongly influenced, in what regards autonomous computing systems, by Shannon's information theory and by the works of Turing and of Von Neumann in computer science. Turing distinguished himself particularly in the definition of an automatic computing machine and a universal computing machine (Turing, 1936), as well as in the roots of artificial intelligence (Turing, 1950). Von Neumann proposed an automaton theory and introduced the basis for the scientific field of artificial life (von Neumann, 1963; Langton, 1995).

These works influenced different scientific disciplines and authors, one in particular, Hugh Everett III, in his PhD thesis on quantum mechanics (Everett, 1973), was the first to introduce cybernetics notions to address the foundations of quantum theory, namely, by using information theory's notions to address quantum theory and by introducing a notion of observer

defined, in Everett's proposal, as an automaton with a memory register capable of interacting with the quantum system and becoming entangled with it (Everett, 1973; 1957).

Despite this early work, quantum cybernetics, as a subfield within cybernetics, is mainly the result of the advancement of research in: quantum computation theory and quantum information theory; complex quantum systems science and quantum game theory, during the 1970s, 1980s and 1990s.

The roots of quantum computation and quantum information theory can be traced back to the work of Everett himself in his application of information theory to the framework of quantum mechanics and his proposal of a quantum automaton model to deal with the observer and the observation act. Other roots can be found in several works during the 1970s and 1980s, namely, in the 1970s one can find, for instance, the work of Weisner on quantum coding, in the research field of quantum cryptography, which also led to quantum game theory (Wiesner, 1983; Meyer, 1999; Piotrowski and Śładkowski, 2003), and Holevo's work on quantum information (Holevo, 1973), as well as other authors like Feynman (1982), Beniof (1982) and Deutsch (1985), who moved forward the research field of quantum computation. Deutsch, in particular, introduced the notion of a universal quantum computing machine based upon a quantum extension of Turing's machine. The research field of quantum computation advanced considerably in terms of potential practical applications with Shor's factoring algorithm and Grover's search algorithm (Shor, 1994; Grover, 1996 ;Nielsen and Chuang, 2010).

Today, quantum computation theory is a growing multidisciplinary field that intercrosses quantum theory with computer science and the complexity sciences (in particular complex quantum systems science). The intersection of quantum computation and quantum game theory, on the other hand, opens the way for research on autonomous artificial systems, incorporating quantum artificial intelligence, as well as for research in the growing field of quantum biology<sup>2</sup>

<sup>2</sup>University of Surrey's Institute of Advanced Studies (IAS), hosted a conference on quantum biology, jointly organized by IAS, the Biotechnology and Biological Sciences Research Council (BBSRC) and MILES (Models and Mathematics in Life and Social Sciences). The following link contains more information on the conference including the presentations:  
<http://www.ias.surrey.ac.uk/workshops/quantumbiology/>



(Löwdin, 1963; McFadden and Al-Khalili, 1999; McFadden, 2000; Ball, 2011). Quantum cybernetics deepens the link between complex quantum systems science and (quantum) computer science, with strong connections to the work developed on the foundations of quantum theory and the role of information in quantum mechanics.<sup>3</sup>

Applications of quantum adaptive computation and quantum optimization also extends to both quantum biology research and quantum game theory, in the case of the later as an adaptive computing framework within interdisciplinary fields such as econophysics, with proven effectiveness in dealing, for instance, with financial risk modeling (Gonçalves, 2011; 2013).

The current work addresses another direction within quantum cybernetics, which is the extension to the quantum setting of the connectionist approach to artificial intelligence, regarding the artificial neural networks' research (McCulloch and Pitts, 1943; Rosenblatt, 1957, 1958; Dupuy, 2000; Müller *et al.*, 1995). In the current article, three main points of cybernetics are, thus, addressed and interlinked in regards to the quantum setting: the notion of autonomy in complex computing systems; quantum artificial intelligence and quantum artificial neural networks.

### 3. Quantum Computation and Quantum Artificial Intelligence

The main insight behind quantum computation regards the representation of quantum information, while the most basic unit of classical information can be expressed in terms of a binary digit called "bit", which is either zero or one, the most basic unit of quantum information can be expressed in terms of a quantum binary digit or "qubit". The major difference is that the qubit corresponds to a superposition of the classical information states. In quantum computation theory, the qubit is addressed, formally, as a vector of a two-dimensional Hilbert space  $H_2$  with computational basis  $\{|0\rangle, |1\rangle\}$ , a qubit, using Dirac's *bra-ket* notation for vectors, is then defined as a normalized ket vector:

$$|\psi\rangle = \psi(0)|0\rangle + \psi(1)|1\rangle \quad (1)$$

where the weights correspond to complex amplitudes satisfying the normalization condition  $|\psi(0)|^2 + |\psi(1)|^2 = 1$ .

Quantum computations on a single qubit can, thus, be addressed in terms of the group  $U(2)$ , the unitary group in 2 dimensions, whose elements correspond to quantum gates, using the quantum circuit terminology (Nielsen and Chuang, 2010). If we assume the following matrix representation of the computational basis:

$$|0\rangle \equiv \begin{pmatrix} 1 \\ 0 \end{pmatrix}, |1\rangle \equiv \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad (2)$$

the general unitary operator can be shown to have the following matrix representation:

$$\hat{U}_{\varphi_0, u, v} = e^{i\varphi_0} \begin{pmatrix} u & v \\ -v^* & u^* \end{pmatrix} \quad (3)$$

with the normalization condition  $|u|^2 + |v|^2 = 1$ .

Thus, we have a global phase transformation  $e^{i\varphi_0}$  (an element of the  $U(1)$  group) and a unitary  $2 \times 2$  matrix with unit determinant (an element of the  $SU(2)$  group). The element of  $SU(2)$  can be expressed as:

$$\begin{pmatrix} u & v \\ -v^* & u^* \end{pmatrix} = \begin{pmatrix} e^{i\varphi_1} \cos \varphi_3 & e^{i\varphi_2} \sin \varphi_3 \\ -e^{-i\varphi_2} \sin \varphi_3 & e^{-i\varphi_1} \cos \varphi_3 \end{pmatrix} \quad (4)$$

so that, introducing the tuple  $\Phi \equiv (\varphi_0, \varphi_1, \varphi_2, \varphi_3)$ , we can rewrite, for the  $U(2)$  matrix:

$$\hat{U}_{\Phi} = e^{i\varphi_0} \begin{pmatrix} e^{i\varphi_1} \cos \varphi_3 & e^{i\varphi_2} \sin \varphi_3 \\ -e^{-i\varphi_2} \sin \varphi_3 & e^{-i\varphi_1} \cos \varphi_3 \end{pmatrix} \quad (5)$$

In the matrix representation, different operators  $\hat{U}_{\Phi}$  can be addressed, with respect to the parameter range, under the notion of matrix identity, indeed, working from the notion of matrix identity, the  $U(1)$  component is such that  $e^{i(\varphi_0 + 2\pi)} = e^{i\varphi_0}$ , so that any two quantum gates differing in the  $U(1)$  component by a phase of  $2\pi$  coincide, thus, the angle  $\varphi_0$  can be defined in the interval  $[0, 2\pi]$ , on the other hand, still working with matrix identity, for a given  $\varphi_0$ , the phases for each entry also share this periodicity so that  $\varphi_1$  and  $\varphi_2$  return the corresponding entry to the same element with a  $2\pi$  periodicity, leading to the interval  $[0, 2\pi]$  for both of these angles as well. Now considering the three angles fixed

<sup>3</sup>An early work addressing these issues is Seth Lloyd's (1988).  
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$\varphi_0, \varphi_1, \varphi_2$ , we are left with the problem of addressing a parametrization taking into account the trigonometric functions, there are two trigonometric functions appearing in the matrix entries:  $\sin \varphi_3$  and  $\cos \varphi_3$ .

The range for the angle  $\varphi_3$  must be assumed, in this case, between 0 and  $\pi/2$  for the reason that for  $\varphi_3 > \pi/2$ , the element of U(2) can be written in terms of a matrix with an angle between 0 and  $\pi/2$  with a different phase, which means that the entire system of quantum gates is covered by  $\varphi_3$  taken in the interval  $[0, \pi/2]$ . As way to make all of the angles range in the same interval  $[0, 2\pi]$ , we have to rewrite the matrix as:

$$\hat{U}_\varphi = e^{i\varphi_0} \begin{pmatrix} e^{i\varphi_1} \cos \frac{\varphi_3}{4} & e^{i\varphi_2} \sin \frac{\varphi_3}{4} \\ -e^{-i\varphi_2} \sin \frac{\varphi_3}{4} & e^{-i\varphi_1} \cos \frac{\varphi_3}{4} \end{pmatrix} \quad (6)$$

assuming, now, also the range  $\varphi_3 \in [0, 2\pi]$ .

This is equivalent to the polar representation of a 3-sphere, so that  $e^{i\varphi_1} \cos \varphi_3 / 4$  and  $e^{i\varphi_2} \sin \varphi_3 / 4$ , with  $\varphi_1, \varphi_3 \in [0, 2\pi]$  correspond to a reparametrization of the usual spherical coordinates: usually, the angle  $\varphi_3$  for the trigonometric functions is assumed as ranging in 0 to  $\pi$  and the division is set by 2 instead of 4, however, it is useful to address the quantum decision dynamics of an AQCS, by letting all angles range from 0 to  $2\pi$ , which explains the division by 4, this due to a simplification of the Hamiltonian formulation for the U(2) group wave function, a point to which we will return further on.

As it stands, from the above representation, it follows that quantum computation can be addressed in terms of quantum gates acting on qubits, such that, in the quantum circuit model representation, information travels through the “wires” and is transformed by the “gates” (Nielsen and Chuang, 2010). This type of representation corresponds to a model of a programmable quantum computing machine, with the quantum algorithm defined in terms of the unitary transformations (quantum gates).

However, if we wish to address quantum artificial intelligence (quantum AI) we need to allow for autonomous quantum computation

defining an AQCS, which means that the quantum computing system must be capable of adapting to different environmental conditions and respond accordingly, thus, given an input qubit  $|\psi_{in}\rangle$  we can consider that for different states of an environment  $|f\rangle$  the quantum AI must be able to evaluate this environment responding with a different quantum program synthesized in the form of a quantum gate applied to  $|\psi_{in}\rangle$ , in this sense, we obtain a quantum conditional response to the environment in the form of a conditional quantum computation with the alternative quantum computation histories  $|f\rangle \otimes \hat{U}_f |\psi_{in}\rangle$ . A specific case is the one where each alternative quantum computation is performed, with a given amplitude:

$$|\Psi\rangle = \int d^4\varphi \Psi(\varphi) |\varphi\rangle \otimes \hat{U}_\varphi |\psi_{in}\rangle \quad (7)$$

where  $\Psi(\varphi)$  is a quantum amplitude over the U(2) parameter space, thus, for the above parameterization, it is only different from zero for  $\varphi_k \in [0, 2\pi]$ , with  $k=0,1,2,3$ , thus covering each of the alternative U(2) elements only once, under the above parametrization of U(2). The ket  $|\Psi\rangle$  is an expression of a quantum AI adapting to a quantum environment which takes the form of an interaction that takes place coupling to the U(2) transformation. Instead of a single quantum program, the quantum computation depends upon this interaction, in this sense, we can speak of a quantum cognition, since the quantum AI responds to the different environmental states with a different quantum computation. We are dealing above with a form of quantum hypercomputation, in the sense that the system (environment+AQCS) leads to a (quantum) superposition of the alternative quantum computations on the qubit.

One may also introduce a Hamiltonian unitary evolution for the environment, which assumes that the U(2) quantum computation is related to a quantum fluctuating environment that is described by a Hamiltonian description of the U(2) group, using the parameter space, under the above parametrization, for the definition, which leads to the change of (7) to the following structure:

$$|\Psi(t)\rangle = \int d^4\varphi e^{-i\hat{H}t} \Psi(\varphi) |\varphi\rangle \otimes \hat{U}_\varphi |\psi_{in}\rangle \quad (8)$$

Due to the parametrization of the operators  $\hat{U}_\varphi$ , where  $\varphi$  is a four-tuple





$(\varphi_0, \varphi_1, \varphi_2, \varphi_3)$ , with each angle  $\varphi_k$  between 0 and  $2\pi$ , one possible choice of Hamiltonian may consider addressing the Hamiltonian in terms of a box potential with the shape of a tesseract with each side measuring  $2\pi$  as the basic structure for the box (a four-dimensional box), however, given the periodic nature of the quantum gates in  $U(2)$ , for the  $U(2)$  parametrization chosen, it follows that we should assume periodic boundary conditions for the tesseract leading to a hypertorus (in this case, a 4-torus). Considering the free Hamiltonian yields the  $U(2)$  connection expressed in equation (8), so that we have a free  $U(2)$  Schrödinger equation, under the previous parametrization, with periodic boundary conditions on the hypertorus:

$$-\nabla^2 \Psi = \lambda \Psi \quad (9)$$

Separating the wave functions:

$$\Psi(\varphi_0, \varphi_1, \varphi_2, \varphi_3) = \Psi(\varphi_0) \Psi(\varphi_1) \Psi(\varphi_2) \Psi(\varphi_3) \quad (10)$$

we can solve the four free Schrödinger equations with the general form:

$$-\frac{d^2}{d\varphi_k} \Psi(\varphi_k) = \lambda_k \Psi(\varphi_k) \quad (11)$$

assuming the periodic condition  $\Psi(\varphi_k + 2\pi) = \Psi(\varphi_k)$ , the solution to this last equation is:

$$\Psi_n(\varphi_k) = \frac{1}{\sqrt{2\pi}} e^{in\varphi_k} \quad (12)$$

with the momentum eigenvalues:

$$n = 0, \pm 1, \pm 2, \dots \quad (13)$$

Letting  $\mathbf{n} = (n_0, n_1, n_2, n_3)$ , the energy eigenvalues are:

$$\lambda_{\mathbf{n}} = n_0^2 + n_1^2 + n_2^2 + n_3^2 \quad (14)$$

So that we have the final eigenfunctions:

$$\begin{aligned} \Psi_{\mathbf{n}}(\boldsymbol{\varphi}) &= \langle \boldsymbol{\varphi} | \mathbf{n} \rangle = \prod_{k=0}^3 \frac{1}{\sqrt{2\pi}} e^{in_k \varphi_k} \\ &= \frac{1}{4\pi^2} \exp\left(i \sum_{k=0}^3 n_k \varphi_k\right) \end{aligned} \quad (15)$$

The reason for the parametrization chosen is now made clear, in the sense that the Hamiltonian condition and the wave function simplify greatly, leading to a free Schrödinger  $U(2)$  equation being expressed as a free Schrödinger equation on a hypertorus (equation (9)).

For equation (8) we can now write:

$$|\Psi(t)\rangle = \int d^4 \boldsymbol{\varphi} \Psi(\boldsymbol{\varphi}, t) |\boldsymbol{\varphi}\rangle \otimes \hat{U}_{\boldsymbol{\varphi}} |w_{in}\rangle \quad (16)$$

where  $\Psi(\boldsymbol{\varphi}, t)$  is the wave packet given by:

$$\Psi(\boldsymbol{\varphi}, t) = \sum_{\mathbf{n}} A_{\mathbf{n}} e^{-i\lambda_{\mathbf{n}} t} \Psi_{\mathbf{n}}(\boldsymbol{\varphi}) \quad (17)$$

with the usual normalization condition assumed:

$$\sum_{\mathbf{n}} |A_{\mathbf{n}}|^2 = 1 \quad (18)$$

Replacing (15) in (17) we get the following solution for the wave packet:

$$\Psi(\boldsymbol{\varphi}, t) = \sum_{\mathbf{n}} A_{\mathbf{n}} \frac{1}{4\pi^2} \exp\left[i\left(\sum_{k=0}^3 n_k \varphi_k - \lambda_{\mathbf{n}} t\right)\right] \quad (19)$$

Taking into account this framework, we are now ready to introduce a quantum version of artificial neural networks based upon the AQCS model. We will deal, in particular, with basic neural connections and QuANNs.

#### 4. Quantum Artificial Neural Networks as Autonomous Quantum Computing Systems

The connectionist paradigm of computation is a strong example of the intersection between computer science and the cybernetics' paradigmatic base. In 1943, McCulloch and Pitts wrote an article entitled *A Logical Calculus of the Ideas Immanent in Nervous Activity*, which introduced another computing framework based on the functioning of neurons, marking the birth of artificial neural networks, now a classic of early cybernetics (Dupuy, 2000). McCulloch and Pitts' proposal was the subject of intense debate within what later came to be the connectionist-based paradigm of computation (Dupuy, 2000; Lafontaine, 2004).

Artificial neural networks (ANNs), while largely based upon a simplification of the human neural structure, form a basic setting for research on network-based computation resulting from the agencing of a large number of elementary computing units called neurons (Müller *et al.*, 1995). ANNs constitute a good mathematical framework for addressing computing network systems as complex computing systems, allowing one to study the relation between the network's architecture and its function.

Information in ANNs has a specific interpretation as well: while in Turing's computational framework, information was addressed in terms of symbols placed in a



machine, in ANNs' computational framework, information is associated with a pattern of activity, the system (the network) is the entity, the system and the pattern of activity are, thus, primitive and fundamental, information being the result of the system's activity and not the "thing" to be printed on a tape (as takes place in Turing's machine), even the memory register can be addressed in terms of the network's recollecting activity (Müller *et al.*, 1995; Dupuy, 2000). In this sense, ANNs are closer to systems science and to a biological paradigm (Müller *et al.*, 1995; Dupuy, 2000), in some way incorporating a bionics<sup>4</sup> base of research.

Quantum artificial neural networks (QuANNs) constitute the quantum extension of ANNs. QuANNs are complex quantum computing systems and allow one to harness the computing power of "networked" quantum computation.

Research on the role of quantum effects in biological neurodynamics has been a topic of interdisciplinary research within neuroscience and physics, with authors like Eccles and Beck, Penrose and Hameroff that defend the hypothesis of quantum neurodynamics in the explanation of consciousness, while other authors, like Tegmark have laid arguments against that hypothesis (Eccles, 1994; Beck and Eccles, 1998; Hameroff and Penrose, 2003; Penrose and Hameroff, 2011; Tegmark, 2000), Tegmark's argument relies on environmental decoherence associated to an interaction with the environment that effectively destroys locally the off-diagonal terms in a subsystem's density matrix, which implies a local entropy increase. However, for this decoherence to take place the system+environment have to become entangled, which means that we actually have macro-coherence with subsystem decoherence.

We will see, in the present section, that entanglement can take place in a neural network itself so that local neuron-level decoherence is almost inevitable, but one cannot state, in this case, that quantum effects are not present, parallel quantum processing in large networks tends to produce entangled network states, in

which the local node description, or even several nodes' description shows local decoherence.

However, QuANNs are not necessarily restricted to research on quantum neurodynamics, or even dependent upon the hypothesis of biological quantum neurodynamics, rather, QuANNs are a model of network-based quantum computation, constituting a framework for addressing parallel quantum computation which can have multiple applications when one deals with complex quantum systems that involve interacting components. QuANNs may also come to play a role in future quantum technologies, points to which we shall return in the last section of the present work.

The connection between quantum theory and artificial neural network theory has been developed, since the 1990s, in connection to quantum computation, in particular, linked to quantum associative memory, parallel processing and schemes of extension of ANNs to the quantum setting (Chrisley, 1995; Kak, 1995; Menneer and Narayanan, 1995; Behrman, *et al.*, 1996; Menneer, 1998; Ivancevic and Ivancevic, 2010).

Considering an artificial neuron as a computing unit with two states 0 (non-firing) and 1 (firing), which, in itself a far simpler model than the biological neuron, allows for network-based classical computation, exploring common aspects with biological neural networks. The quantum extension of McCulloch and Pitts proposal leads one to address the artificial neuron's two states as the basis states of the qubit, which means that the artificial neuron can be in a superposition of the non-firing state, represented by the *ket* vector  $|0\rangle$ , and the firing state, represented by the *ket* vector  $|1\rangle$ .

Extending the quantum circuit approach to quantum computation, in the parallel processing framework, means that we assign qubits to neurons and quantum gates to the synaptic connections, the quantum gates for the synaptic connections express the interaction between the neurons in the form of a quantum computation, which means that the synaptic connection is more complex than in classical ANNs, in the sense that it plays an active role in allowing for network-based quantum computation, including the possibility of adaptive quantum computation.

<sup>4</sup>Notion introduced by Jack E. Steele to refer to biologically-based design of artificial and intelligent systems. Bionics was defended by von Foerster, who greatly influenced the complexity sciences (Asaro, 2007), mainly in what regards biology-based computation approaches. In this sense, ANNs are already concurrent with bionics, as biology-inspired directions within the paradigmatic base of cybernetics that go beyond a mechanistic approach (Dupuy, 2000; Lafontaine, 2004).



To explore the range of possibilities we address, in the current work, quantum feedforward neural networks. In its simplest form the (classical) feedforward neural network consists of two layers of neurons: a first layer of input neurons and a second layer of output neurons (Müller *et al.*, 1995). The neurons of the output layer receive synaptic signals from the input layer. A more complex model includes multiple layers, allowing for more complex computations (Müller *et al.*, 1995). Quantum feedforward neural networks can be used to compute all classical Boolean functions in a more efficient way than classical feedforward neural networks, as we now show.

#### 4.1. Quantum feedforward neural networks and Boolean function computation

We will begin by addressing, in this section, examples of application of quantum feedforward neural networks, in the examples we always start with the network such that each neuron is in the non-firing state  $|0\rangle$ , the quantum computation proceeds then as a form of quantum learning dynamics through quantum parallel processing. In the examples, we will consider first the computation of Boolean functions of the form  $\{0,1\}^m \rightarrow \{0,1\}$ , and afterwards generalize to the Boolean functions  $\{0,1\}^m \rightarrow \{0,1\}^n$ . There are, for general  $m$ ,  $2^{2^m}$  such Boolean functions, considering  $m=1$ , as a first illustrative example, we get the four basic Boolean functions:

**Table 1.** Boolean functions  $\{0,1\} \rightarrow \{0,1\}$ .

$g(s)$	$s=0$	$s=1$
$g(s)=0$	0	0
$g(s)=s$	0	1
$g(s)=1-s$	1	0
$g(s)=1$	1	1

The QuANN that is capable of computing these Boolean functions is a two-layered quantum feedforward network:

$$N_1 \rightarrow N_2 \quad (20)$$

where  $N_1$  is the input neuron and  $N_2$  is the output neuron. The initial state of the two neurons is  $|00\rangle$  (both are non-firing), following the previous section's approach, if we deal with

the above network as an AQCS, the ket vector for the network's quantum computation histories' ket is given by:

$$|\Psi\rangle = \int d^4\boldsymbol{\varphi} \Psi(\boldsymbol{\varphi}) |\boldsymbol{\varphi}\rangle \otimes \hat{N}_{\boldsymbol{\varphi}} |00\rangle \quad (21)$$

where  $\hat{N}_{\boldsymbol{\varphi}}$  is an operator on the Hilbert space  $H_2 \otimes H_2$ , corresponding to a single quantum computation history for the quantum network, and  $\Psi(\boldsymbol{\varphi})$  is the quantum amplitude associated with each alternative history, in this case taken as a wave packet of the form:

$$\Psi(\boldsymbol{\varphi}) = \sum_n A_n \Psi_n(\boldsymbol{\varphi}) \quad (22)$$

such that the Hamiltonian condition, represented by the Schrödinger equation (9), becomes a Hamiltonian restriction for the network's quantum computation. The structure of  $\hat{N}_{\boldsymbol{\varphi}}$  is such that the input neuron undergoes a quantum fluctuation due to an interaction with the environment and the output neuron plays the role of a generalized Everett automaton that interacts with the input neuron through the synaptic connection, more complex quantum computations will be considered in the next section. Given this structure, in regards to the Boolean function computation,  $\hat{N}_{\boldsymbol{\varphi}}$  takes the form:

$$\hat{N}_{\boldsymbol{\varphi}} = \hat{S}_g \hat{U}_{\boldsymbol{\varphi}} \otimes \hat{1} \quad (23)$$

thus,  $\hat{N}_{\boldsymbol{\varphi}}$  is such that a U(2) transformation is first applied to the input neuron leaving the output neuron unchanged (operator  $\hat{U}_{\boldsymbol{\varphi}} \otimes \hat{1}$ ) (this corresponds to the interaction between the input neuron and the environment) and, then,  $\hat{S}_g$  is applied, representing the neural network's internal networked processing, where  $\hat{S}_g$  is a unitary operator on the Hilbert space  $H_2 \otimes H_2$ , whose structure depends upon the Boolean function under computation. In the neural network model, this operator expresses the structure of the synaptic connection which determines the interaction pattern between the two neurons. The following table shows the general structure for the synaptic connection quantum gates  $\hat{S}_g$  and the corresponding alternative output kets associated with each quantum computation histories:

**Table 2.** Output kets for the neural network computing the Boolean functions  $\{0,1\} \rightarrow \{0,1\}$ .

$g(s)$	$\hat{S}_g$	Output Kets
$g(s)=0$	$\hat{1} \otimes \hat{1}$	$ \psi_{\phi,0}\rangle$
$g(s)=s$	$ 0\rangle\langle 0  \otimes \hat{1} +  1\rangle\langle 1  \otimes \hat{U}_{NOT}$	$\psi_{\phi}(0) \phi,00\rangle$ $+ \psi_{\phi}(1) \phi,11\rangle$
$g(s)=1-s$	$ 0\rangle\langle 0  \otimes \hat{U}_{NOT} +  1\rangle\langle 1  \otimes \hat{1}$	$\psi_{\phi}(0) \phi,01\rangle$ $+ \psi_{\phi}(1) \phi,10\rangle$
$g(s)=1$	$\hat{1} \otimes \hat{U}_{NOT}$	$ \psi_{\phi,1}\rangle$

In the first case,  $\hat{S}_g$  has the structure:

$$\hat{S}_g = \sum_{s=0}^1 (|s\rangle\langle s| \otimes \hat{1}) = \hat{1} \otimes \hat{1} \quad (24)$$

therefore, the synaptic connection is neutral, in the sense that the activation pattern of the input neuron in nothing affects the activation pattern of the output neuron, thus, for an initial non-firing configuration for the two neurons  $|00\rangle$ , the quantum computation histories are given by:

$$\hat{N}_{\phi} = (\hat{1} \otimes \hat{1})(\hat{U}_{\phi} \otimes \hat{1}) = \hat{U}_{\phi} \otimes \hat{1} \quad (25)$$

which means that the input neuron's state undergoes the unitary transformation  $\hat{U}_{\phi}$ , but it does not affect the output neuron (neutral synaptic connection), thus, the output neuron remains non-firing, independently of the input neuron, so that replacing (25) in (21) we obtain:

$$|\Psi\rangle = \int d^4\phi \Psi(\phi) |\phi, \psi_{\phi,0}\rangle \quad (26)$$

where we took  $|\phi, \psi_{\phi,0}\rangle = |\phi\rangle \otimes |\psi_{\phi}\rangle \otimes |0\rangle$ , with  $|\psi_{\phi}\rangle = \psi_{\phi}(0)|0\rangle + \psi_{\phi}(1)|1\rangle$ , for the amplitudes resulting from the U(2) transformation:

$$\psi_{\phi}(0) = \exp[i(\varphi_0 + \varphi_1)] \cos(\varphi_3 / 4)$$

$$\text{and } \psi_{\phi}(1) = -\exp[i(\varphi_0 - \varphi_2)] \sin(\varphi_3 / 4).$$

For the second and third cases, the output neuron functions similarly to a quantum memory register, so that the system has an incorporated Everett-like automaton structure. For the second case,  $\hat{S}_g$  has the structure of a controlled-not gate<sup>5</sup>:

$$\hat{S}_g = |0\rangle\langle 0| \otimes \hat{1} + |1\rangle\langle 1| \otimes \hat{U}_{NOT} = \hat{U}_{CNOT(1)} \quad (27)$$

where  $\hat{U}_{NOT}$  is the unitary "NOT" gate:

$$\hat{U}_{NOT} = |0\rangle\langle 1| + |1\rangle\langle 0| \quad (28)$$

The synaptic connection, in this case, corresponds to a reinforcing synaptic connection, thus, for an initial non-firing configuration for the two neurons  $|00\rangle$ , the quantum computation histories are given by:

$$\hat{N}_{\phi} = \hat{U}_{CNOT(1)} \hat{U}_{\phi} \otimes \hat{1} \quad (29)$$

which means that a quantum fluctuation takes place at the input neuron and then the synaptic interaction with the second neuron leads to an entangled state, in which either both neurons fire or not.

It is important to notice that the entangled state shows a quantum correlation, in the sense that the configuration is either the two neurons fire synchronously without delay or not.

The synaptic connection acts in such a way as to allow an interaction where the second neuron effectively performs a von Neumann measurement upon the first neuron and, thus, acts in a similar way to an Everett automaton's memory register dynamics.

For the third case, the situation is reversed, instead of a controlled-not operator on  $|1\rangle$ , the controlled negation uses  $|0\rangle$  as the control:

$$\hat{S}_g = |0\rangle\langle 0| \otimes \hat{U}_{NOT} + |1\rangle\langle 1| \otimes \hat{1} = \hat{U}_{CNOT(0)} \quad (30)$$

the synaptic connection, in this case, is inhibitory. As in the previous case a quantum fluctuation takes place at the input neuron and the synaptic interaction with the second neuron leads to an entangled state, however, in this case, if one of the neurons is firing the other is not and vice-versa. Replacing in (21), the second and third cases lead, respectively, to the kets:

$$|\Psi\rangle = \int d^4\phi \Psi(\phi) (\psi_{\phi}(0)|\phi,00\rangle + \psi_{\phi}(1)|\phi,11\rangle) \quad (31)$$

$$|\Psi\rangle = \int d^4\phi \Psi(\phi) (\psi_{\phi}(0)|\phi,01\rangle + \psi_{\phi}(1)|\phi,10\rangle) \quad (32)$$

In the last case, the quantum computation histories are of the kind:

<sup>5</sup>We write the controlled-not  $\hat{U}_{CNOT(1)}$  with the subscript  $CNOT(1)$  instead of the usual notation  $CNOT$ , this is done

because we will work, below, with another controlled not operator that performs the negation when the control qubit is  $|0\rangle$  rather than  $|1\rangle$ , this other operator will be denoted as  $\hat{U}_{CNOT(0)}$ .





$$\hat{N}_{\phi} = \hat{1} \otimes \hat{U}_{NOT} \hat{U}_{\phi} \otimes \hat{1} \quad (33)$$

this means that while the input neuron is placed in a superposition, the second neuron fires no matter what happens to the input neuron, this can be considered as a global transformation of both neurons that lead to a superposition state for the input neuron and to the firing of the output neuron, the network's quantum computation histories' ket is, in this case, given by:

$$|\Psi\rangle = \int d^4\phi \Psi(\phi) |\phi, \psi_{\phi} 1\rangle \quad (34)$$

From the four cases, for the synaptic connection quantum gate  $\hat{S}_g$ , one can see that its general form can be expressed as follows:

$$\hat{S}_g = \sum_{s=0}^1 |s\rangle\langle s| \otimes \hat{U}_{g(s)} \quad (35)$$

where the operator  $\hat{U}_{g(s)}$  has the following structure:

$$\hat{U}_{g(s)=0} = \hat{U}_0 \equiv \hat{1} \quad (36)$$

$$\hat{U}_{g(s)=1} = \hat{U}_1 \equiv \hat{U}_{NOT} \quad (37)$$

thus if the Boolean function evaluates to 0, then, the operator  $\hat{U}_{g(s)}$  coincides with the unit operator  $\hat{1}$  on the two-dimensional Hilbert space  $H_2$ , since the initial configuration of the network is always set so that the output neuron is not firing,  $\hat{U}_{g(s)=0}$  this means that the output neuron does not fire when the input neuron exhibits the activity pattern  $|s\rangle$ , on the other hand, when the Boolean function evaluates to 1, the operator  $\hat{U}_{g(s)}$  coincides with the NOT quantum gate  $\hat{U}_{NOT}$ , since the initial configuration of the network is always set so that the output neuron is not initially firing, this means that the output neuron fires when the input neuron exhibits the activity pattern  $|s\rangle$ . This can be generalized, in the sense that controlled negations can be used to compute any Boolean function with feedforward QuANNs, as we now show.

Considering, first, the case of Boolean functions  $\{0,1\}^m \rightarrow \{0,1\}$ , the quantum feedforward neural network that is able to compute these functions is comprised of  $m$  input neurons and one output neuron, without need for

hidden layers. The network's quantum computation histories' ket is given by:

$$|\Psi\rangle = \int d^4\phi_m \dots d^4\phi_1 \Psi(\phi_1, \dots, \phi_m) |\phi_1, \dots, \phi_m\rangle \otimes \hat{N}_{\phi_1, \dots, \phi_m} |00\dots 0; 0\rangle \quad (38)$$

where we used, for the ket notation, the semicolon symbol ";" to separate, in the ket, the input neurons' states from the output neuron's state, and the operator  $\hat{N}_{\phi_1, \dots, \phi_m}$  is given by:

$$\hat{N}_{\phi_1, \dots, \phi_m} = \hat{S}_g \left( \bigotimes_{k=1}^m \hat{U}_{\phi_k} \right) \otimes \hat{1} \quad (39)$$

where  $\hat{S}_g$  has the structure:

$$\hat{S}_g = \sum_{s_1 \dots s_m} |s_1 \dots s_m\rangle\langle s_1 \dots s_m| \otimes \hat{U}_{g(s_1 \dots s_m)} \quad (40)$$

such that if  $g(s_1 \dots s_m) = 0$ ,  $\hat{U}_{g(s_1 \dots s_m)} = \hat{1}$  while if  $g(s_1 \dots s_m) = 1$ ,  $\hat{U}_{g(s_1 \dots s_m)} = \hat{U}_{NOT}$ .

In this case, the quantum neural networks grow with the number of input neurons, which reduces the structural complexity of the networks as compared for instance to the Boolean function computation by a classical feed-forward neural network with a hidden layer. For classical feedforward neural networks, the input layer grows with  $m$  and it is possible to compute every Boolean function of the form  $\{0,1\}^m \rightarrow \{0,1\}$  by a scheme that includes a hidden layer of neurons that grows as  $2^m$  (Müller *et al.*, 1995). Quantum feedforward neural networks, on the other hand, do not need the hidden layer to compute these Boolean functions, and grow in proportion to  $m$ .

As stressed by Müller *et al.* (1995) the classical case is a computationally hard problem (NP-complete). The problem is reduced in complexity in the quantum computation setting since the number of nodes does not grow as  $m+2^m$  ( $m$  input neurons plus  $2^m$  hidden neurons), it only grows as  $m$  (the number of input neurons), without a hidden layer. This result can be extended to the Boolean functions  $\{0,1\}^m \rightarrow \{0,1\}^n$ .

In this case, we use the following notation: if  $g$  is a Boolean function such that  $g: \{0,1\}^m \rightarrow \{0,1\}^n$ , then, for an input string

$s_1 s_2 \dots s_m$  if  $g$  evaluates to  $g(s_1 \dots s_m) = s_1' s_2' \dots s_n'$  we define the local mappings  $g_l, l = 1, 2, \dots, n$  as:

$$g_l(s_1 \dots s_m) = s_l' \quad (41)$$

where  $s_l'$  is the  $l$ -th symbol of the output string  $s_1' s_2' \dots s_n'$ . With this notation in place, in order to compute the general Boolean functions  $\{0,1\}^m \rightarrow \{0,1\}^n$ , we need a quantum feedforward neural network with  $m$  input neurons and  $n$  output neurons (once again, the network grows as  $m+n$ ). Assuming that the initial state of all neurons (input and output) is non-firing, the quantum computation histories' ket is given by:

$$|\Psi\rangle = \int d^4 \varphi_m \dots d^4 \varphi_1 \Psi(\varphi_1, \dots, \varphi_m) |\varphi_1, \dots, \varphi_m\rangle \otimes \hat{N}_{\varphi_1, \dots, \varphi_m} |00\dots 0; 00\dots 0\rangle \quad (42)$$

where, once more, we used the semicolon symbol ";" to separate, in the ket, the input neurons' states from the output neurons' states. As before, the computation histories  $\hat{N}_{\varphi_1, \dots, \varphi_m}$  are defined as:

$$\hat{N}_{\varphi_1, \dots, \varphi_m} = \hat{S}_g \left( \bigotimes_{k=1}^m \hat{U}_{\varphi_k} \right) \otimes \hat{1}_n \quad (43)$$

where  $\hat{1}_n$  is the unit operator on the  $2^n$  dimensional Hilbert space for the output neurons' states given by the  $n$  tensor product of copies of  $H_2$ . The operator  $\hat{S}_g$ , in this case, has the structure:

$$\hat{S}_g = \sum_{s_1 \dots s_m} |s_1 \dots s_m\rangle \langle s_1 \dots s_m| \otimes \hat{B}_{g(s_1 \dots s_m)} \quad (44)$$

where  $\hat{B}_{g(s_1 \dots s_m)}$  corresponds to the Boolean function operator defined as:

$$\hat{B}_{g(s_1 \dots s_m)} = \bigotimes_{l=1}^n \hat{U}_{g_l(s_1 \dots s_m)} \quad (45)$$

where the local unitary operators  $\hat{U}_{g_l(s_1 \dots s_m)}$  are defined such that  $\hat{U}_{g_l(s_1 \dots s_m)} = \hat{1}$  when  $g_l(s_1 \dots s_m) = 0$  and  $\hat{U}_{g_l(s_1 \dots s_m)} = \hat{U}_{NOT}$  when  $g_l(s_1 \dots s_m) = 1$ , in this way, the Boolean function operator leads to a firing of each neuron or not, depending upon the Boolean function's local mapping evaluating to either 0 or 1, respectively. Thus, the interaction between the input and output neurons, expressed by the operator  $\hat{S}_g$ , leads to the neural computation of the Boolean functions, so that the

network yields the correct output through the pattern of activity of the output layer. Only two layers are, once more, needed: an input layer of  $m$  input neurons and an output layer of  $n$  output neurons, so that the network grows in size as  $m+n$ .

While two layers allow for Boolean function computation, multiple layers can be used for multiple Boolean function computations as well as more complex neural network configurations, as we now show.

## 4.2 Multiple layers and generalized quantum processing in quantum feedforward neural networks

Let us consider the quantum feedforward neural network with the following architecture:



The second layer allows for the computation of the Boolean functions  $g: \{0,1\} \rightarrow \{0,1\}^2$ , while the third layer allows for the computation of the Boolean functions  $h: \{0,1\}^2 \rightarrow \{0,1\}$ , using the last section's framework, we separate each layer's state by a semicolon, so that initially we have the network configuration where each neuron is not firing, expressed by the ket:

$$|0; 00; 0\rangle \quad (47)$$

The quantum histories for the quantum neural network are given by:

$$\hat{N}_{\varphi} = \hat{S}_h \hat{S}_g \left( \hat{U}_{\varphi} \otimes \hat{1}_3 \right) \quad (48)$$

thus, as before, the unitary gate  $\hat{U}_{\varphi}$  is applied to the input neuron state, leaving the output neurons unchanged, then, the first Boolean function is implemented, represented by the operator  $\hat{S}_g$  (which is associated to the synaptic connections from the first to the second layer) and the second Boolean function is implemented, represented by the third layer's synaptic connections operator  $\hat{S}_h$ , the operators are, respectively, given by:

$$\hat{S}_g = \left( \sum_s |s\rangle \langle s| \otimes \hat{B}_{g(s)} \right) \otimes \hat{1} \quad (49)$$



$$\hat{S}_h = \hat{1} \otimes \left( \sum_{s,s'} |ss'\rangle \langle ss'| \otimes \hat{B}_{h(ss')} \right) \quad (50)$$

with the Boolean operators given by:

$$\hat{B}_{g(s)} = \hat{U}_{g_1(s)} \otimes \hat{U}_{g_2(s)} \quad (51)$$

$$\hat{B}_{h(ss')} = \hat{U}_{h(ss')} \quad (52)$$

If, for instance, the functions are defined as:

$$g(0) = 0, g(1) = 1 \quad (53)$$

$$h(ss) = 0, h(s1 - s) = 1 \quad (54)$$

The quantum computation histories' ket, resulting from the above equations, is given by:

$$|\Psi\rangle = \int d^4 \boldsymbol{\varphi} \Psi(\boldsymbol{\varphi}) |N_{\boldsymbol{\varphi}}\rangle \quad (55)$$

with  $|N_{\boldsymbol{\varphi}}\rangle \equiv \psi_{\boldsymbol{\varphi}}(0) |\boldsymbol{\varphi}, 0; 01; 1\rangle + \psi_{\boldsymbol{\varphi}}(1) |\boldsymbol{\varphi}, 1; 10; 1\rangle$ .

Therefore, in this case, the second neuron becomes entangled with the input, through a reinforcing synaptic connection, while the third neuron becomes entangled with the input neuron through a inhibitory synaptic connection, which means that the two neurons' states in the middle layer are no longer separable, they share a quantum correlation with the input and, thus, with each other, such that the first two layers correspond to a fully entangled QuANN. The fourth neuron fires if the second neuron fires and the third does not or if the second neuron does not fire and the third fires, but it does not fire if both the second and third neurons fire or if both the second and third neurons do not fire, which is in accordance with the fact that  $\hat{S}_h$  is computing the exclusive-or Boolean function at the third layer of the neural network.

Since the second and third neurons share an inhibitory quantum correlation it follows that the fourth neuron always fires. In this way, the fourth neuron allows one to introduce a level of reflexivity regarding the neural network's activity, in the sense that the fourth neuron fires with the computation of a specific neural configuration in the middle layer. It is, thus, possible to build "awareness" neurons that is: neurons that fire when certain patterns in the neural network take place, and do not fire otherwise. Crossing levels, one could even introduce another link from the input neuron  $N_1$  to  $N_4$ , and make the computation conditional on the specific patterns of the network up to that point.

While the efficient computation of Boolean functions is a central point of the application of quantum feedforward neural networks, one can go beyond the Boolean logical calculus and define more general synaptic quantum gates.

Thus, for instance, for the same network architecture (46), if the operators  $\hat{U}_{h(ss')}$  in equation (52) are replaced by the following operators:

$$\hat{U}_{H,h(ss')} = \hat{U}_H \hat{U}_{h(ss')} \quad (56)$$

where  $\hat{U}_H$  is the quantum Haddamard transform:

$$\hat{U}_H = \frac{1}{\sqrt{2}} (|0\rangle\langle 0| - |1\rangle\langle 1| + |0\rangle\langle 1| + |1\rangle\langle 0|) \quad (57)$$

and the Boolean function is replaced by:

$$h(00) = h(11) = 0 \quad (58)$$

$$h(01) = 0, h(10) = 1 \quad (59)$$

then, for this new sequence of quantum computations,  $\hat{N}_{\boldsymbol{\varphi}}$  leads to the following transformation:

$$\begin{aligned} |N_{\boldsymbol{\varphi}}\rangle &\equiv \hat{N}_{\boldsymbol{\varphi}} |0; 00; 0\rangle = \\ &= \psi_{\boldsymbol{\varphi}}(0) |0; 01\rangle \otimes |+\rangle + \psi_{\boldsymbol{\varphi}}(1) |1; 10\rangle \otimes |-\rangle \end{aligned} \quad (60)$$

with  $|+\rangle = \hat{U}_H |0\rangle$  and  $|-\rangle = \hat{U}_H |1\rangle$ , thus:

$$|\pm\rangle = \frac{1}{\sqrt{2}} (|0\rangle \pm |1\rangle) \quad (61)$$

The final ket, for the quantum computation histories, is no longer (55) but the following:

$$\begin{aligned} |\Psi\rangle &= \int d^4 \boldsymbol{\varphi} \Psi(\boldsymbol{\varphi}) |N_{\boldsymbol{\varphi}}\rangle = \\ &= \int d^4 \boldsymbol{\varphi} \Psi(\boldsymbol{\varphi}) \\ &(\psi_{\boldsymbol{\varphi}}(0) |\boldsymbol{\varphi}, 0; 01\rangle \otimes |+\rangle + \psi_{\boldsymbol{\varphi}}(1) |\boldsymbol{\varphi}, 1; 10\rangle \otimes |-\rangle) \end{aligned} \quad (62)$$

which means that, for each alternative branch, the last neuron is always in a superposition between firing and non-firing. Even though the last neuron is effectively entangled with the rest of the network, the entanglement is for the alternative basis (61), in which the neuron does not have a well defined firing pattern, showing, rather, a superposition between two alternative firing patterns.

A quantum feedforward neural network can yield such superpositions in the sense that it

can generate a complementarity between firing patterns, in the sense that if a neuron is in a well defined firing activity, then, the other is in a superposition between firing and not firing and vice-versa, this is a specifically quantum effect. For classical ANNs this cannot take place. While (62) provides for a more complex example, the simplest example can be obtained for a two-layered quantum feedforward neural network with the architecture (20). With the following structure for  $\hat{N}_\varphi$ :

$$\hat{N}_\varphi = \hat{S}_{12} \hat{U}_\varphi \otimes \hat{1} \quad (63)$$

$$\hat{S}_{12} = |0\rangle\langle 0| \otimes (\hat{U}_H \hat{1}) + |1\rangle\langle 1| \otimes (\hat{U}_H \hat{U}_{NOT}) \quad (64)$$

which leads to the result:

$$|\Psi\rangle = \int d^4\varphi \Psi(\varphi) (\psi_\varphi(0) |\varphi, 0, +\rangle + \psi_\varphi(1) |\varphi, 1, -\rangle) \quad (65)$$

therefore, in the branch where the input neuron does not fire, the output neuron is in a superposition between firing and not firing described by  $|+\rangle$ , similarly, in the branch where the input neuron fires, the output neuron is in a superposition between firing and not firing described by  $|-\rangle$ , for each alternative neural computing history, the input neuron is entangled with the output neuron, only it is not entangled with respect to the basis  $\{|0\rangle, |1\rangle\}$ , but, rather, with respect to the basis  $\{|+\rangle, |-\rangle\}$ , which means that the first neuron and the second neuron share a similarity to the position and momentum in quantum mechanics, expressed by the notion of quantum complementarity, if the input is in a well defined firing pattern (either not firing or firing) the output no longer has a well defined firing pattern being in a neural state of superposition between firing and not firing.

## 5. Complex Quantum Systems and Quantum Connectionism

In terms of technological development, so far, we have not yet reached the point of building a QuANN, so that QuANNs largely constitute a theoretical ground for exploring the properties of complex quantum computing systems, and provide for a relevant line for the expansion of quantum cybernetics in dialogue with complex quantum systems science.

The quantum connectionist framework of QuANNs can be understood as a general

framework for dealing with “networked” quantum computation, and not necessarily limited to the quantum neural processing conjecture. In different fields of application of quantum theory one can find the possible usefulness of incorporating QuANNs as conceptual tools for addressing self-organization at the quantum level, for instance, Zizzi, in her quantum computational approach to loop quantum gravity, reached a development point that can be identified as an intersection between spin networks and QuANNs (Zizzi, 2003; Zizzi, 2005).

Considering QuANNs as complex quantum computing systems we can see, from the examples of the quantum feedforward neural networks addressed in the present work, that once we allow for an ANN to support networked quantum computation through quantum superposition of neural firing patterns and through quantum gates expressing synaptic connectivity quantum dynamics, new properties start to emerge, that are not present for classical ANNs: the system is capable of quantum hypercomputation, it is possible to compute efficiently all the Boolean functions without over-increasing the number of neurons, and new entangled states are possible where the network may exhibit complementarity between entangled neurons, such that if a neuron is firing the other is in a superposition of neural states.

The examples, addressed in the current work, show how QuANNs may have highly entangled states and can lead to a greater complexity in terms of quantum computation. Three major points were introduced here that are directly connected to quantum cybernetics, namely: the issue of autonomy; the issue of hypercomputation and the issue of reflexivity associated with quantum neural computation.

To go from a standard programmable quantum computer to a complex quantum computing system, we introduced the notion of an autonomous quantum computing system (AQCS), such that the system has an incorporated artificial intelligence that allows it to choose from different alternative quantum circuits in response to an interaction with a quantum environment that couples to the U(2) gates, the interaction between system and environment is, in this case, adaptive (that is, it triggers a quantum computation) and leads to a simultaneous evaluation of each alternative quantum





computation history. In the case of QuANNs, this framework is complexified by the fact that the entire network's quantum computation is responsive to the environment, namely the interaction with the environment triggers the input neurons, which in turn triggers the synaptic interactions allowing for complex responses, the entire network's (quantum computational) activity is, in this way, responsive to the environment, so that the alternative quantum computations  $\hat{N}_{\Phi_1, \Phi_2, \dots, \Phi_m}$  that are implemented with corresponding amplitudes  $\Psi(\Phi_1, \Phi_2, \dots, \Phi_m)$  synthesize a level of reflexive response to the quantum environment.

In the case of the quantum feedforward neural networks, the operators  $\hat{N}_{\Phi_1, \Phi_2, \dots, \Phi_m}$  lead to quantum computations that can cross the different layers of the feedforward neural network, so that the whole network is computing the initial neural stimulus. This is a form of hypercomputation in the sense that alternative quantum computations can be performed with different amplitudes throughout the neural network.

The network, thus, shows a level of distributed reflexivity in the sense that the whole of the network reflects in a complex way the initial neural stimulus that leads to the quantum fluctuation of the input neurons. In this distributed reflexivity not only can the network possess neurons that fire when they identify certain quantum neural patterns (leading to a level of network-based reflexivity), but the network itself may show a level of diversity in the neural patterns, exploring the quantum complementarity for different neurons, thus exhibiting complex configurations that can only take place once we go to the quantum computation framework.

The possibility to solve NP-hard problems of network-based computation, with a smaller amount of computing resources is one of the points that may support some researchers' conjecture of quantum biocomputation in actual biological systems, which is already supported by empirical evidence (McFadden, 2000). In what regards neurobiology, this entails the possibility of a form of quantum biological neural processing, however, as previously stated, the community is still divided on this last issue.

In this sense, the main value of QuANNs is mostly as mathematical and physical models for network-based quantum computation, both QuANNs as well as ANNS can be approached mathematically as general models of computing networks, a point, therefore, remains, as explored in the present work, once we have network-based quantum computation and examples of biological systems exploring quantum effects, we are led to quantum networked computation, and, in this case, the framework of QuANNs may provide for a tool to explore such cases independently of what the nodes represent (actual neurons or any other active quantum processing). Thus, there is high value of QuANNs as conceptual tools for both applied research on quantum (bio)technology using network-based quantum computation<sup>6</sup>, as well as for fundamental research on quantum biology, neuroscience and complex quantum systems science.

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<sup>6</sup>We are referring here to both hybrid technologies (with incorporated biological components) as well as biologically-inspired technologies. For instance, the research on the role of quantum mechanics in photosynthesis may lead to more efficient solar energy-based technologies. Also, in a more advanced state of quantum technologies, the possibility of implementing evolutionary quantum computation based on QuANNs may lead to more adaptive artificial intelligent systems with higher computing power and smaller processing times, yielding faster and more efficient responses (quantum-based agility solutions for intelligent systems).



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