Ramp Heating Response In Thermoelastic Thick Annular Disc With Convective Heat Exchange Boundaries Under Fractional Order Derivatives

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Abstract

The focus of this study is to determine the thermoelastic behaviour of thick annular disc subjected to ramp heating on the upper face while convective heat exchange boundary conditions are prescribed on the curved surfaces. The heat conduction equation involves the Caputo type nonlocal time fractional derivative of order α . The solution of heat conduction equation is obtained in Laplace transform domain and its inversion is done by utilizing the Gaver–Stehfest approach. For the numerical analysis aluminum material is considered and computed results for the temperature distribution, displacement function and thermal stresses are plotted graphically for weak, normal and strong conductivity.

Keywords: Annular disc, fractional order theory of thermoelasticity, thermal stress, integral transform

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1. Introduction

Thermal analysis plays an important role in design of various structural materials which are highly used for industrial purpose and having great applications in science and engineering field. Lots of significant studies from the long time have been going on the analysis of temperature and stress distribution by considering different thermoelastic objects. Kulkarni and Deshmukh [5] discuss the steady state thermoelastic problem of a thick disc with thermally insulated edges by Hankel transform method. Kulkarni and Deshmukh [6] determine thermal stresses in a thick annular disc having arbitrary heat flux at lower and upper surface with fixed circular edge by integral transformation method. Jalali and Shahriari [7] analyze elastic stresses in a functionally graded rotating disc with variable rotating thickness. Shinde, Navlekar and Ghadle [8] formulate inverse heat conduction problem of annular disc and determine temperature distribution and stress function.

Lamba and Khobragade [9] studied thermoelastic problem of a rectangular object and determine its thermal stresses by the theory of integral transformation. Kamdi and Lamba [10] analyze the displacement function and thermal stresses in a functionally graded hollow cylinder with uniform temperature field. Zenkour and Mashat [11] presented the both exact and numerical solution of rotating variable thickness annular disc. Roy Choudhury [12] solved the quasi static thermal deflection in a thin circular plate due to the action of ramp-type heating.

In the study of mathematical models based on fractional calculus it is noticed that they predicts retardation response, not instantaneous once, which found more accord in many physical observations. This retardation response is found absent in classical modeling. So the fractional calculus acts as a powerful tool to explain physical process.
Caputo and Mainardi [13] and Caputo [14] employ fractional derivative to analyze the

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dissipation model and successfully discussed experimental results for anelastic solids.

Povstenko [15] investigate thermal stress for one and two dimensional fractional heat conduction problems. Povstenko [16] determined stresses in one dimensional axisymmetric case and solved nonhomogeneous telegraph equation with fractional order.

Sherief, El-Sayed and A. M. A. El-Latief [18] derived new fractional order theory by employing fractional calculus methodology. Ezzat and Fayik [19] presented thermo-diffusion theory based on concept of fractional calculus also derived uniqueness and reciprocity theorems. Raslan [20] solved two dimensional thick problems in application of fractional thermoelasticity. Raslan [21] discuss one dimensional problem in generalized medium for a spherical shell by integral transform method.

Sherief and AbdEl-Latief [22] determine fractional parameter behaviour of half space problem with variable thermal conductivity under fractional thermoelasticity. Sherief and AbdEl-Latief [23] investigated temperature, displacement and stress behaviour in half space one dimensional problem under thermal shock in context of fractional order thermoelasticity theory. Ezzat, El-Karamany and El-Bary [24] discussed a new model based on thermo-viscoelasticity in isotropic media with relaxation operator of fractional order and solve half space problem with ramp heating. Lamba [25] analyze the temperature and thermal response of a functionally graded cylinder with temperature dependent material properties under the fractional order theory of thermoelasticity. Thakare, Warbhe and Lamba [26] solved nonhomogeneous thermoelastic problem of thick hollow cylinder with convective boundaries and heat generation under fractional order thermoelastic approach. Kamdi and Kumar [27] analyze the thermal behaviour under fractional order derivative for a problem of an annular fin. Kumar and Kamdi [28] discuss the behaviour of two-dimensional axisymmetric hollow cylinder with convective boundaries and internal heat source as a liner function of temperature. Mahdy et al. [29] determine analytically the effect of magneto-photothermal problem with time fractional derivative with Thomson effect and initial stress. Bayones et al. [30] discuss heat conduction model of thermoelastic thin strip with temperature dependent thermal conductivity and thermal shock. Xu, Wang and Xue [31] determine the fractional thermal stresses in a material subjected to short pulse laser heating and discuss its thermo-physics behaviour in Laplace transform domain. Povstenko et al. [32] studied an infinite space problem of an external circular crack under axisymmetric heat flux loading within the framework of fractional thermoelasticity. Recently, Lamba and Roy [33] analytically solved the thermoelastic problem of circular sector disk under the influence of fractional theory with internal heat generation.

From the available literature it is noted that till date not so much work on thermoelastic thick bodies has been done within the context of fractional thermoelasticity subjected to ramp heating and convective heat exchange boundary conditions. Hence, to overcome the gap of research in this field the authors are highly motivated to study and analyze thermoelastic behaviour of thick annular disc subjected to ramp heating on the upper face while convective heat exchange boundary conditions on curved surfaces under fractional thermoelasticity.

2. Problem formulation

We consider a time fractional thermoelastic thick annular disc of order $0 < \alpha \le 2$, with thickness $2h$, internal radius a and external radius b, occupying the space $D = \{(x, y, z) \in R^3 : a \le (x^2 + y^2)^{1/2} \le b, -h \le z \le h\}$,
Where $r = (x^2 + y^2)^{1/2}$. Let convective heat exchange boundary conditions on the inner and outer curved surfaces of thick disc is described and subjected to temperature distribution at the upper surface $z = h$ as follows

$$[T]_{z=h} = \frac{\theta_0}{t_0}\{H(r) - H(r - r_0)\}t \quad ; 0 \le t \le t_0$$
$$= \theta_0 \{H(r) - H(r - r_0)\} \quad ; t \ge t_0$$

In which $f(r) = \{H(r) - H(r - r_0)\}$ is the difference in Heaviside function, t is time, T_0 defines the reference temperature distribution which does not produce stress or strain in the plate, and t_0 is a fixed ramp parameter value, respectively.

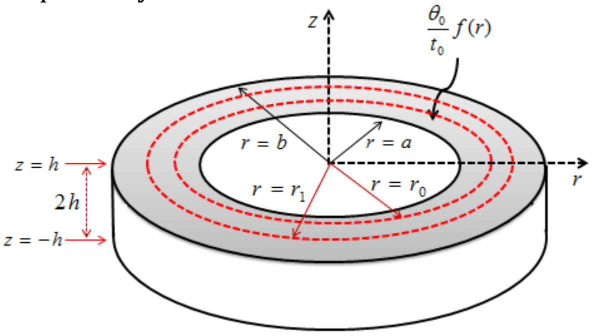

Fig.1. Geometry of the problem

2.1 Temperature distribution function

The transient heat conduction equation with internal heat generation is given as

$$\frac{\partial^2\theta}{\partial r^2}+\frac{1}{r}\frac{\partial\theta}{\partial r}+\frac{\partial^2\theta}{\partial z^2}=\frac{1}{\kappa}\frac{\partial^\alpha\theta}{\partial t^\alpha}\ ;$$

$$a\le r\le b,\ -h\le z\le h \qquad (1)$$

Where, $\frac{\partial^\alpha\theta}{\partial t^\alpha}$ denotes the Caputo type nonlocal time fractional derivative of temperature function θ w.r.t. t of order α as [17]

$$\frac{d^\alpha\theta(t)}{dt^\alpha}=\begin{cases}\frac{1}{\Gamma(n-\alpha)}\int_0^t (t-\tau)^{n-\alpha-1}\frac{d^n\theta(\tau)}{d\tau^n}d\tau, & n-1<\alpha<n,\\ \frac{d^n\theta(\tau)}{dt^n}, & \alpha=n\end{cases} \qquad (2)$$

Where $\kappa=\lambda/\rho C$, λ being the thermal conductivity of the material of the disc, ρ is the density and C is the calorific capacity, assumed to be constant.

$$\theta=0 \quad \text{at } t=0,\ 0<\alpha\le 2, \qquad (3a)$$

$$\frac{\partial\theta}{\partial t}=0 \ \text{ at } t=0,\ 1<\alpha\le 2, \qquad (3b)$$

$$\theta-k_1\frac{\partial\theta}{\partial r}=0,\ \text{ at } r=a,\ -h\le z\le h\ ,\ t>0 \qquad (3c)$$

$$\theta+k_2\frac{\partial\theta}{\partial r}=0,\ \text{at } r=b,\ -h\le z\le h\ ,\ t>0 \qquad (3d)$$

$$\theta=0 \text{ at } z=-h,\ a\le r\le b\ ,\ t>0 \qquad (3e)$$

$$\theta=\begin{cases}\left(\frac{\theta_0}{t_0}\right)f(r)t\ ; & 0\le t\le t_0\\ \theta_0 f(r)\ ; & t>t_0\end{cases},\quad \text{at } z=h,\ a\le r\le b\ ,\ t>0 \qquad (3f)$$

2.2 Displacements and thermal stress components

The Navier's equations in the absence of body forces for axisymmetric two-dimensional thermoelastic problem can be expressed as [4]

$$\nabla^2 u_z-\frac{1}{1-2\upsilon}\frac{\partial e}{\partial z}-\frac{2(1+\upsilon)}{1-2\upsilon}\alpha_t\frac{\partial\theta}{\partial z}=0 \qquad (4)$$

$$\nabla^2 u_r-\frac{u_r}{r}+\frac{1}{1-2\upsilon}\frac{\partial e}{\partial r}-\frac{2(1+\upsilon)}{1-2\upsilon}\alpha_t\frac{\partial\theta}{\partial r}=0 \qquad (5)$$

Where u_r and u_z are the displacement components in the radial and axial directions, respectively and the dilatation e as

$$e=\frac{\partial u_r}{\partial r}+\frac{u_r}{r}+\frac{\partial u_z}{\partial z} \qquad (6)$$

The displacement function in the cylindrical coordinate system are represented by the Goodier's thermoelastic displacement potential ϕ and Love's function L as [1]

$$u_r=\frac{\partial\phi}{\partial r}-\frac{\partial^2 L}{\partial r\,\partial z}, \qquad (7)$$

$$u_z=\frac{\partial\phi}{\partial z}+2(1-\upsilon)\nabla^2 L-\frac{\partial^2 L}{\partial^2 z} \qquad (8)$$

In which Goodier's thermoplastic potential must satisfy the equation

$$\nabla^2\phi=\left(\frac{1+\upsilon}{1-\upsilon}\right)\alpha_t\theta \qquad (9)$$

and the Love's function L must satisfy the equation

$$\nabla^2(\nabla^2 L)=0 \qquad (10)$$

Where, $\nabla^2=\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{\partial^2}{\partial z^2}$

The component of the stresses are represented by the use of the potential ϕ and Love's function L as

$$\sigma_{rr}=2G\left\{\left(\frac{\partial^2\phi}{\partial r^2}-\nabla^2\phi\right)+\frac{\partial}{\partial z}\left(\upsilon\nabla^2 L-\frac{\partial^2 L}{\partial r^2}\right)\right\}, \qquad (11)$$

$$\sigma_{\theta\theta}=2G\left\{\left(\frac{1}{r}\frac{\partial\phi}{\partial r}-\nabla^2\phi\right)+\frac{\partial}{\partial z}\left(\upsilon\nabla^2 L-\frac{1}{r}\frac{\partial L}{\partial r}\right)\right\}, \qquad (12)$$

$$\sigma_{zz} = 2G\left\{\left(\frac{\partial^2\phi}{\partial r^2} - \nabla^2\phi\right) + \frac{\partial}{\partial z}\left((2-\upsilon)\nabla^2 L - \frac{\partial^2 L}{\partial z^2}\right)\right\}, \quad (13)$$

and

$$\sigma_{rz} = 2G\left\{\frac{\partial^2\phi}{\partial r\partial z} + \frac{\partial}{\partial r}\left((1-\upsilon)\nabla^2 L - \frac{\partial^2 L}{\partial z^2}\right)\right\} \quad (14)$$

Where G and υ are the shear modulus and Poisson's ratio respectively.
The boundary condition on the traction free surface functions are

$$\sigma_{rz}\big|_{r=a} = \sigma_{rz}\big|_{r=b} = 0 \quad (15)$$

Equations (1) to (15) constitute the mathematical formulation of the problem.

3. Solution of the problem

3.1 Solution of the heat conduction problem

In order to solve equation (1) under the boundary condition (2a) and (2b), we firstly define the Laplace transform of time fraction order derivative of order α over the variable t as [17]

$$L\left\{\frac{d^\alpha\theta(r,z,t)}{dt^\alpha}\right\} = s^\alpha\theta^*(r,z,s) - \sum_{k=0}^{n-1}\theta^{(k)}(0^+)s^{\alpha-1-k}, \quad n-1<\alpha<n \quad (16)$$

$$L\{\theta(r,z,s)\} = \theta^*(r,z,s)$$

in which s is the transform parameter.
Applying the transform defined in (16), to the equations (1) and utilizing boundary conditions (3a) to (3f) we get

$$\frac{\partial^2\theta^*}{\partial r^2} + \frac{1}{r}\frac{\partial\theta^*}{\partial r} + \frac{\partial^2\theta^*}{\partial z^2} = \frac{1}{\kappa}s^\alpha\theta^* \;;\; a \le r \le b, -h \le z \le h \quad (17)$$

$$\theta^* - k_1\frac{\partial\theta^*}{\partial r} = 0, \text{ at } r = a, -h \le z \le h, t > 0 \quad (18)$$

$$\theta^* + k_2\frac{\partial\theta^*}{\partial r} = 0, \text{ at } r = b, -h \le z \le h, t > 0 \quad (19)$$

$$\theta^* = 0 \text{ at } z = -h, a \le r \le b, t > 0 \quad (20)$$

$$\theta^* = \left(\frac{\theta_0}{t_0}\right)f(r)\left(\frac{1-e^{-st_0}}{s^2}\right), \text{ at } z = h, a \le r \le b, t > 0 \quad (21)$$

θ^* is the Laplace transformed function of θ and s is transformed parameter.
Now, we assume the temperature distribution is given by

$$\theta^*(r,z,s) = \frac{\pi}{\sqrt{2}}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}A_m\frac{\beta J_0(\beta_m b)Y_0(\beta_m b)}{\left[1-\frac{J_0^2(\beta_m b)}{J_0^2(\beta_m a)}\right]^{\frac{1}{2}}}\left[\frac{J_0(\beta_m r)}{J_0(\beta_m b)} - \frac{Y_0(\beta_m r)}{Y_0(\beta_m b)}\right]\sinh(\chi_n(z+h)) \quad (22)$$

And β_m's are the positive roots of transcendental

$$\left[\frac{J_0(\beta a)}{J_0(\beta b)} - \frac{Y_0(\beta a)}{Y_0(\beta b)}\right] = 0 \quad (23)$$

Above assumed temperature function (22) successfully satisfied the boundary condition (18) to (20), now to satisfy the boundary condition (21), assume the Fourier-Bessel series as

$$f(r) = \sum_{m=1}^{\infty}B_m J_0(\beta_m r) \quad (24)$$

Then from the theory of Bessel function

$$B_m\int_a^b r[J_0(\beta_m r)]^2 dr = \int_a^b r\, f(r)J_0(\beta_m r)dr = \int_{r_0}^{r_1} r\, J_0(\beta_m r)dr \quad (25)$$

since

$$f(r) = \begin{cases} 1 \;; & r_0 \le r \le r_1 \\ 0 \;; & a \le r \le r_0,\; r_1 \le r \le b \end{cases}$$

Hence

$$B_m = \frac{2}{\beta_m}\left\{\frac{r_1 J_1(\beta_m r_1) - r_0 J_1(\beta_m r_0)}{b^2[J_0(\beta_m b)]^2 - a^2[J_0(\beta_m a)]^2}\right\} \quad (26)$$

Substituting Eq. (22) into Eq. (17), one obtains

$$\chi_n^2 = \frac{s^\alpha}{\kappa} + \zeta_{mn}^2 \quad (27)$$

$$\zeta_{mn}^2 = \left(\gamma_n^2 + \beta_m^2\right) \quad (28)$$

Substituting Eq. (26) into the boundary condition (21) for $z = h$, one obtains

$$A_m = \frac{2}{\beta_m}\frac{\theta_0}{t_0}\left\{\frac{r_1 J_1(\beta_m r_1) - r_0 J_1(\beta_m r_0)}{b^2[J_0(\beta_m b)]^2 - a^2[J_0(\beta_m a)]^2}\right\}\frac{1}{\sin(\chi_n h)}\left(\frac{1-e^{-st_0}}{s^2}\right) \quad (29)$$

By replacing the values of Eq. (29) into Eq. (22), one obtains

$$\theta^*(r,z,s) = \sqrt{2}\,\pi\frac{\theta_0}{t_0}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}K_0(\beta_m, r)$$

$$\frac{1}{\beta_m}\left\{\frac{r_1 J_1(\beta_m r_1) - r_0 J_1(\beta_m r_0)}{b^2[J_0(\beta_m b)]^2 - a^2[J_0(\beta_m a)]^2}\right\}\frac{1}{\sinh(\chi_n h)}$$

$$\times\left(\frac{1-e^{-st_0}}{s^2}\right)\times\sinh\left(\left[\frac{s^\alpha}{\kappa} + \zeta_{mn}^2\right]^{\frac{1}{2}}(z+h)\right) \quad (30)$$

Where

$$K_0(\beta_m, r) = \frac{\beta J_0(\beta_m b) Y_0(\beta_m b)}{\left[1 - \frac{J_0^2(\beta_m b)}{J_0^2(\beta_m a)}\right]^{\frac{1}{2}}} \left[\frac{J_0(\beta_m r)}{J_0(\beta_m b)} - \frac{Y_0(\beta_m r)}{Y_0(\beta_m b)}\right]$$

3.2 Solution of the thermal stress problem

Referring to the fundamental equation (1) and its solution (30) for the heat conduction problem, the solution for the displacement function are represented by the Goodier's thermoelastic displacement potential ϕ governed by equation (9) are represented in the Laplace domain as

$$\phi^*(r,z,s) = \sqrt{2}\left(\frac{1+\upsilon}{1-\upsilon}\right)\alpha_t \frac{\theta_0}{t_0} \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{K_0(\beta_m, r)}{(\gamma_n^2 + \beta_m^2)} \frac{1}{\beta_m}$$

$$\left\{\frac{r_1 J_1(\beta_m r_1) - r_0 J_1(\beta_m r_0)}{b^2[J_0(\beta_m b)]^2 - a^2[J_0(\beta_m a)]^2}\right\} \frac{1}{\sinh(\chi_n h)}$$

$$\times\left(\frac{1-e^{-st_0}}{s^2}\right) \times \sinh\left(\left[\frac{s^\alpha}{\kappa} + \zeta_{mn}^2\right]^{\frac{1}{2}}(z+h)\right) \qquad (31)$$

Similarly, the solution for Love's function L^* are assumed so as to satisfy the governed condition of equation (10) as

$$L^*(r,z,s) = \sqrt{2}\left(\frac{1+\upsilon}{1-\upsilon}\right)\alpha_t \frac{\theta_0}{t_0} \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{K_0(\beta_m, r)}{(\gamma_n^2 + \beta_m^2)} \frac{1}{\beta_m}$$

$$\left\{\frac{r_1 J_1(\beta_m r_1) - r_0 J_1(\beta_m r_0)}{b^2[J_0(\beta_m b)]^2 - a^2[J_0(\beta_m a)]^2}\right\} \frac{1}{\sinh(\chi_n h)}$$

$$\times[\sinh(\beta_m z) + z\cosh(\beta_m z)]\left(\frac{1-e^{-st_0}}{s^2}\right) \times \sinh\left(\left[\frac{s^\alpha}{\kappa} + \zeta_{mn}^2\right]^{\frac{1}{2}}(z+h)\right) \qquad (32)$$

Now, in order to obtain the displacement components represented in the Laplace domain, we substitute the values of thermoelastic displacement potential ϕ^* and Love's function L^* in equations (7) and (8), one obtains

$$u_r^*(r,z,s) = \sqrt{2}\left(\frac{1+\upsilon}{1-\upsilon}\right)\alpha_t \frac{\theta_0}{t_0} \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{K_0'(\beta_m, r)}{(\gamma_n^2 + \beta_m^2)} \frac{1}{\beta_m}$$

$$\left\{\frac{r_1 J_1(\beta_m r_1) - r_0 J_1(\beta_m r_0)}{b^2[J_0(\beta_m b)]^2 - a^2[J_0(\beta_m a)]^2}\right\} \frac{1}{\sinh(\chi_n h)}$$

$$\times[-(\beta_m + 1)\cosh(\beta_m z) - \beta_m z \sinh(\beta_m z)]$$

$$\left(\frac{1-e^{-st_0}}{s^2}\right) \times \sinh\left(\left[\frac{s^\alpha}{\kappa} + \zeta_{mn}^2\right]^{\frac{1}{2}}(z+h)\right) \qquad (33)$$

$$u_z^*(r,z,s) = \sqrt{2}\left(\frac{1+\upsilon}{1-\upsilon}\right)\alpha_t \frac{\theta_0}{t_0} \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{K_0(\beta_m, r)}{(\gamma_n^2 + \beta_m^2)} \frac{1}{\beta_m}$$

$$\left\{\frac{r_1 J_1(\beta_m r_1) - r_0 J_1(\beta_m r_0)}{b^2[J_0(\beta_m b)]^2 - a^2[J_0(\beta_m a)]^2}\right\} \frac{1}{\sinh(\chi_n h)}$$

$$\times[-\gamma_n(Q_n \sin(\gamma_n z) + W_n \cos(\gamma_n z)) - 4\upsilon\beta_m$$

$$\sin(\beta_m z) - \beta_m^2(\sinh(\beta_m z) + z\cosh(\beta_m z))]$$

$$\times\left(\frac{1-e^{-st_0}}{s^2}\right) \times \sinh\left(\left[\frac{s^\alpha}{\kappa} + \zeta_{mn}^2\right]^{\frac{1}{2}}(z+h)\right) \qquad (34)$$

Thus, making use of the two displacement components, the dilation is established represented in the Laplace domain as

$$e^* = \sqrt{2}\left(\frac{1+\upsilon}{1-\upsilon}\right)\alpha_t \frac{\theta_0}{t_0} \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{K_0(\beta_m, r)}{(\gamma_n^2 + \beta_m^2)} \frac{1}{\beta_m}$$

$$\left\{\frac{r_1 J_1(\beta_m r_1) - r_0 J_1(\beta_m r_0)}{b^2[J_0(\beta_m b)]^2 - a^2[J_0(\beta_m a)]^2}\right\} \frac{1}{\sinh(\chi_n h)}$$

$$\times[(\beta_m + 1)\cosh(\beta_m z) + \beta_m z \sinh(\beta_m z)$$

$$-\ \gamma_n^2 - (4\nu + 1)\ \beta_m^2 \cosh(\beta_m z)]$$

$$\times\left(\frac{1-e^{-st_0}}{s^2}\right) \times \sinh\left(\left[\frac{s^\alpha}{\kappa} + \zeta_{mn}^2\right]^{\frac{1}{2}}(z+h)\right) \qquad (35)$$

Then, the stress components represented in the Laplace domain can be evaluated by substituting the values of thermoelastic displacement potential ϕ^* and Love's function L^* from equation (31) and equation (32) in equations (11), (12), (13) and (14), one obtains

$$\sigma_{rr}^* = 2\sqrt{2}\, G\left(\frac{1+\upsilon}{1-\upsilon}\right)\alpha_t \frac{\theta_0}{t_0} \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{K_0(\beta_m, r)}{(\gamma_n^2 + \beta_m^2)} \frac{1}{\beta_m}$$

$$\left\{\frac{r_1 J_1(\beta_m r_1) - r_0 J_1(\beta_m r_0)}{b^2[J_0(\beta_m b)]^2 - a^2[J_0(\beta_m a)]^2}\right\} \frac{1}{\sinh(\chi_n h)}$$

$$\times\left\{\left(\beta_m^2 + \gamma_n^2\right)^{-1} K_0''(\beta_m, r) + K_0(\beta_m, r)\right\} - \left(\beta_m^2 + \gamma_n^2\right)^{-1}\left[2\nu\ \beta_m^2 \cosh(\beta_m z)\right.$$

$$\times K_0(\beta_m,r)-[(\beta_m+1)\cosh(\beta_m z)+z\beta_m\sinh(\beta_m z)]K_0''(\beta_m,r)]\}$$
$$\times\left(\frac{1-e^{-st_0}}{s^2}\right)\times\sinh\left(\left[\frac{s^\alpha}{\kappa}+\zeta_{mn}^2\right]^{\frac{1}{2}}(z+h)\right) \quad (36)$$

$$\sigma_{\theta\theta}^* = 2\sqrt{2}\,G\left(\frac{1+\upsilon}{1-\upsilon}\right)\alpha_t\frac{\theta_0}{t_0}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{-K_0(\beta_m,r)}{(\gamma_n^2+\beta_m^2)}\frac{1}{\beta_m}$$
$$\left\{\frac{r_1J_1(\beta_m r_1)-r_0J_1(\beta_m r_0)}{b^2[J_0(\beta_m b)]^2-a^2[J_0(\beta_m a)]^2}\right\}\frac{1}{\sinh(\chi_n h)}$$
$$\times\left\{-\left[r(\beta_m^2+\gamma_n^2)^{-1}K_0'(\beta_m,r)+K_0(\beta_m,r)\right.\right.$$
$$\left.\right]-(\beta_m^2+\gamma_n^2)^{-1}[2\nu\ \beta_m^2\cosh(\beta_m z)$$
$$\times\left(\frac{1-e^{-st_0}}{s^2}\right)\times\sinh\left(\left[\frac{s^\alpha}{\kappa}+\zeta_{mn}^2\right]^{\frac{1}{2}}(z+h)\right) \quad (37)$$

$$\sigma_{zz}^* = 2\sqrt{2}\,G\left(\frac{1+\upsilon}{1-\upsilon}\right)\alpha_t\frac{\theta_0}{t_0}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}-K_0(\beta_m,r)$$
$$\frac{1}{\beta_m}\left\{\frac{r_1J_1(\beta_m r_1)-r_0J_1(\beta_m r_0)}{b^2[J_0(\beta_m b)]^2-a^2[J_0(\beta_m a)]^2}\right\}\frac{1}{\sinh(\chi_n h)}$$
$$\times\left\{-\left[(\beta_m^2+\gamma_n^2)^{-1}K_0''(\beta_m,r)+K_0(\beta_m,r)\right]\right.$$
$$\left.\right]+[(2\nu+\beta_m)\ \beta_m^2\cosh(\beta_m z)$$
$$\times\left(\frac{1-e^{-st_0}}{s^2}\right)\times\sinh\left(\left[\frac{s^\alpha}{\kappa}+\zeta_{mn}^2\right]^{\frac{1}{2}}(z+h)\right) \quad (38)$$

$$\sigma_{rz}^* = 2\sqrt{2}\,G\left(\frac{1+\upsilon}{1-\upsilon}\right)\alpha_t\frac{\theta_0}{t_0}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}-\frac{-K_0(\beta_m,r)}{(\gamma_n^2+\beta_m^2)}$$
$$\frac{1}{\beta_m}\left\{\frac{r_1J_1(\beta_m r_1)-r_0J_1(\beta_m r_0)}{b^2[J_0(\beta_m b)]^2-a^2[J_0(\beta_m a)]^2}\right\}\frac{1}{\sinh(\chi_n h)}$$
$$\times\{\gamma_n[Q_n\sin(\gamma_n z)+W_n\cos(\gamma_n z)]K_0(\beta_m,r)+[2\nu\beta_m\sin(\gamma_n z)$$
$$+\beta_m^2[\sinh(\gamma_n z)+z\cos(\gamma_n z)]]K_0'(\beta_m,r)\}$$
$$\times\left(\frac{1-e^{-st_0}}{s^2}\right)\times\sinh\left(\left[\frac{s^\alpha}{\kappa}+\zeta_{mn}^2\right]^{\frac{1}{2}}(z+h)\right) \quad (39)$$

4. Inversion of Laplace transforms:

In many problems it is not possible to evaluate the analytical solution of problem due to complexity in inversion of Laplace transform, so method of numerical inversion is adopted here as suggested by Gaver-Stehfast algorithm [34, 35 and 36]. A detailed explanation and the processor can be seen in Knight and Raiche [37]. Gaver and Stehfast modified formula is as follows

$$f(t)=\frac{\ln 2}{t}\sum_{j=1}^{H}C(j,H)\,F\left(j\frac{\ln 2}{t}\right) \quad (40)$$

with

$$C(j,H)=(-1)^{j+M}\sum_{n=m}^{\min(j,M)}\frac{n^M(2n)!}{(M-n)!(n-1)!(j-n)!(2n-j)!}$$

Here, the value of even integer H is dependent of word length of assumed computer programme. $M=H/2$, m is the integer part of the $(j+1)/2$. The value of H is taken from the Stehfast algorithm which converges faster and with desired accuracy. Also to find the involved integrals in the calculations Romberg technique of numerical integration [38] is used subjected to variable step size.

4. Numerical Analysis

For the purpose of numerical computation aluminum metal material properties are considered as:-

a=1 cm, b=1 cm, h=2.00 cm,

Modulus of Elasticity $E = 6.9 \times 10^6$ N/cm^2,

Shear modulus $G = 2.7 \times 10^6$ N/cm^2,

Poisson ratio $\upsilon = 0.281$,

Thermal expansion coefficient, $\alpha_t = 25.5 \times 10^6$ cm/cm-^{0}C,

Thermal diffusivity $\kappa = 0.86$ cm2/sec,

Thermal conductivity $\lambda = 0.48$ cal sec^{-1}/cm ^{0}C

For the mathematical simplicities the radiation coefficients constants are set as:- $k_i=0.86\,(i=1,3)$ and $k_i=1\,(i=2,4)$

In order to examine the influence of ramp heating at the upper surface of thick disc, we performed the numerical calculation for different value of fractional parameters $\alpha=0.5$, $\alpha=1$, $\alpha=1.5\ \&\ \alpha=2$. The resultant distribution of temperature, displacement and thermal stresses along radial direction for weak, normal and strong conductivity is plotted as shown in figures with the help of computer programme.

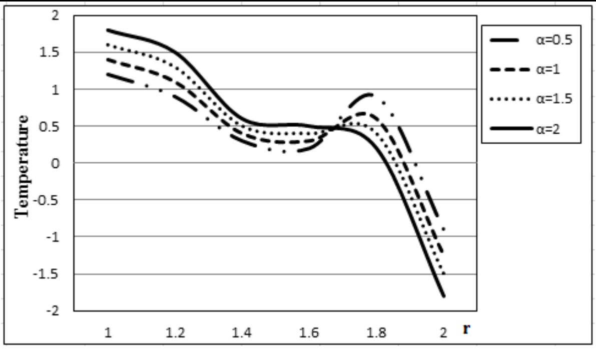

Fig. 2: Temperature distribution

Fig. 2 shows the distribution of temperature along the radial direction for the thick annular disc subjected to ramp heating at surface $z = h$ for different value of fractional parameter $\alpha = 0.5, 1, 1.5, 2$. It is observed that temperature distribution depends on the value fractional parameter α i.e. large the value of fractional parameter higher temperature impact found. Further temperature drop becomes more and more gradually along the radial direction towards $z = 1$.

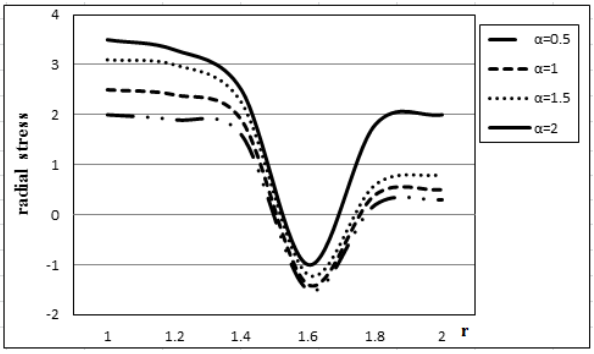

Fig. 3: Radial stress distribution

Fig. 3 shows the thermal stress distribution along the radial direction at the midpoint of the disc. From the figure, it is noted that through the radial direction the location of points of minimum stress occurs at the end points and maximum thermal stress response are pointed at the interior part. Tensile stress behaviour is observed at the inner part and so that outer edges tends to expand more as compare to inner surface. Strong dependency of stress on various fractional parameters is clearly shown from the distribution.

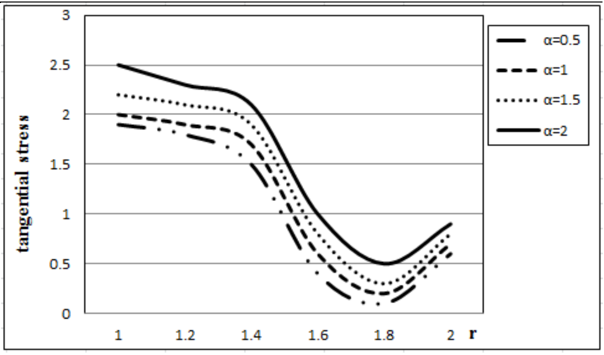

Fig. 4: Tangential stress distribution

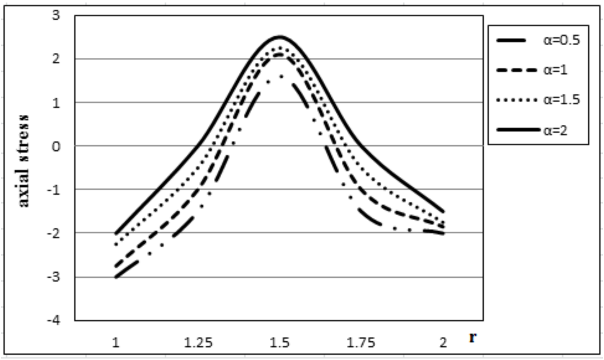

Fig. 5: Axial stress distribution

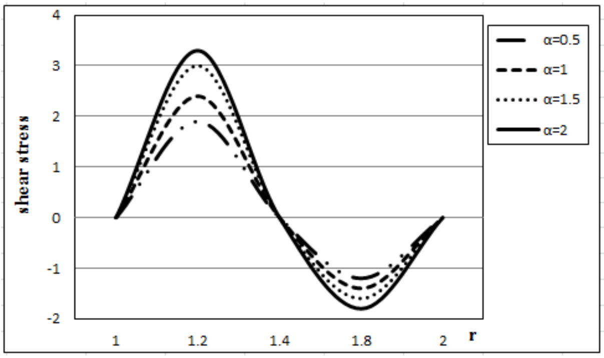

Fig. 6: Shear stress distribution

Fig. 4, 5 and 6 shows the distribution of tangential stress $\sigma_{\theta\theta}$, axial stress σ_{zz} and shear stress σ_{rz} along the radial direction for different fractional parameter. Due to the impact of compressive stresses within the concentric region decreasing trend of tangential stress along the radial direction is reflected. Fig. 4 shows that, the axial stress distribution σ_{zz}, found high at the mid region and of large in magnitude as compared to radial stress component. Also from Fig. 5, sinusoidal nature is observed for shear stress distribution with high crest and troughs along the radial direction of the thick annular disc subjected to ramp heating.

From the numerical analysis, it is predicted that $\alpha = 0.5$ and $\alpha = 1$ curves are monotonic in nature. When $\alpha = 2$ the heat conduction equation predicts finite wave propogation and becomes hyperbolic. Hence, the speed of wave propogation is found clearly affected by changing fraction order parameters. Therefore for purpose of new material designing this prediction can plays an important role which is applicable to real life situations.

Conclusion

In this study, we analyze the thermoelastic behaviour of thick disc with time fractional order derivative under zero initial conditions. By utilizing the concept of Laplace transformation and its inversion technique as suggested by Gaver-Stehfast the expression of temperature variation, displacement function and thermal stresses for a thermoelastic problem of a thick annular disc with ramp heating at the upper surface $z = h$ is obtained successfully. Numerical analysis is done for weak conductivity $(0 < \alpha < 1)$, normal conductivity $(\alpha = 1)$ and strong conductivity $(1 < \alpha < 2)$. As a result retardation response of waves is observed in fractional theory also it can be concluded that adopted system of equations in this study may be useful for the researchers working in field of material sciences for the design of useful structures or machines in engineering applications in the determination of thermoelastic behaviour with radiation using fractional calculus.

Nomenclature

α - is the order of Caputo type time fractional derivative
h - thickness of disc
a - internal radius of disc
b - external radius of disc
$F(r,t)$ - ramp heating at the lower surface of disc
θ - temperature function
λ - being the thermal conductivity of the material of the disc
ρ - density
C - Calorific capacity
$H(r)$ - Heaviside function
u_r - Displacement components in the radial direction
u_z - Displacement components in the axial direction
e – Dilatation
ϕ - Goodier's thermoelastic displacement potential
L - Love's function
G - shear modulus
υ - Poisson's ratio
s - Laplace transform parameter.

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