Original Article

What is The Evidence for Quantum Like Interference Effects in Human Judgments and Decision Behavior?

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Abstract

This article examines the empirical evidence for interference effects in psychological experiments. It also reviews the competing interpretation of these effects with respect to traditional cognitive models and new quantum cognition models. The conclusion is that quantum theory provides unifying principles for explaining interference effects found in a wide variety of different experimental paradigms, and it provides a viable new theoretical approach for understanding cognition and decision making.

Key Words: interference, quantum, Markov, judgment, decision

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Introduction

The target article for this special issue written by Professor Elio Conte reports on evidence for 'quantum like' interference effects in psychology, and this evidence is then used to justify the development of "quantum wave like" representations of the experimental results. What is an interference effect, what is the empirical evidence for these effects, what is the best explanation for these effects, and what direction should we take next? We intend to answer these four questions in this paper. The answers demonstrate that interference effects are widespread in psychology, and quantum models do provide a uniform way to account for these diverse findings, and these promising initial results encourage more intense research *"On the possibility that we think in a quantum probabilistic manner."*

Before beginning we should note that the target article for this special issue raises the possibility for two different types of quantum models for understanding psychology. One is to develop a quantum physical model of the brain, and the other is to develop models that are called 'quantum like' (Khrennikov, 2010) or generalized quantum (Atmanspacher, 2010) or quantum structural (Aerts, 2009) models. The latter are not quantum physics models of the brain, but instead they are mathematical models of human behavior derived from principles abstracted and extrapolated from quantum theory. This article is only concerned with the latter type of theory.

I. What is an interference effect?

Suppose we have two different perceptual judgment tasks: one labeled task A and the response to this task is measured by J different levels of a response variable (e.g. $J=2$ binary forced choice); another is task B with K levels of a response measure (e.g. $K=7$ point confidence rating). Participants are randomly assigned to two groups: one group of participants (group B)

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receives only task A, but the other group (group BA) receives task B followed immediately by task A. (Other variations are of course possible, such as presenting both tasks in different counterbalanced orders, but let us focus on this simple design).

From this experiment we obtain proportions which are estimates of the response probabilities, which includes (a) the probability of choosing level k from the response to task A from group A-alone symbolized as $p(R_A = k)$, (b) the probability of first responding with level j from task B, denoted $p(R_B = j)$, and (c) the probability of responding with level k from task A given that the person responded with level j on the earlier task B, denoted $p(R_A = k \mid R_B = j)$. From the latter two probability distributions we can compute the *total probability* for response to task A as

$$p_T(R_A = k) = \sum_{j=1}^{K} p(R_B = j) \cdot p(R_A = k \mid R_B = j).$$

The interference effect for level k of the response to task A (produced by responding to task B) equals by definition $Int_A(k) = p(R_A = k) - p_T(R_A = k)$.

Note that

$$\sum_{k=1}^{K} p(R_A = k) = 1 = \sum_{k=1}^{K} p_T(R_A = k)$$

so that $\sum_k Int_A(k) = 0$. Interference effects frequently occur in particle physical experiments, such as the findings from the famous two slit experiment. Interference effects with beams of light were first found by Thomas Young which he used as evidence for the wave theory of light. But later these interference effects were obtained even with single particles which prompted the creation of the 'particle - wave' theory of quantum mechanics.

First we simply ask -- do these interference effects occur in human psychology experiments? If they do, there may be many easy explanations such as learning effects and practice and boredom effects, etc. So below we examine studies in which these simple reasons to not seem to exist. The experiments reported below use only $J = 2$ and $K = 2$ levels (e.g. a binary choice is made for each task). In this case we can obtain only one interference effect (the other is the negative of the first).

II. What is the evidence for interference effects?

Below we summarize several lines of research that provide evidence for interference effects. The first three lines described below provide direct tests for interference effects, and the remaining three lines provide indirect tests of interference.

Perception of ambiguous figures

Interference effects were first investigated in the perceptual domain in a preliminary set of studies by Conte et al. (2006), but the later studies using perceptual judgment tasks with ambiguous figures obtained more convincing evidence and we will focus on the latter (Conte, 2009).[2] Each condition in the later study included approximately 100 students randomly divided into two groups: In group A-Alone, each person was given 3 seconds to make a single binary choice concerning an ambiguous figure B (+ indicates one alternative, – indicates the other alternative); in group BA, each person was given 3 seconds to make a single binary choice for an ambiguous figure B followed 800 msec later by a 3 second presentation requesting another single binary choice for figure A. In the first experiment, stimulus A was two horizontal lines of equal length placed in a context creating an illusion that one line was longer than another; stimulus B was two non overlapping circles of equal radius placed in a context creating an illusion that one circle was larger than another; the task was to decide whether the objects were equal or not. In the second experiment, stimulus A was a Rubin type ambiguous figure, and stimulus B was a completely different Rubin type ambiguous figure; and the task was to choose one of the two possible interpretations of the ambiguous figure. The third experiment used a Stroop task with animals of different sizes (e.g., big mouse, small lion) and the task was to choose the larger object. The results are

[2] Actually four experiments were conducted but the second and third only differed by changing the roles of the stimuli assigned to A and B, the results were very similar, and so we collapsed data across these two experiments.

shown in Table 1 below. As can be seen in this table, all three groups produced significant interference effects (using a z test for the difference). Surprisingly the direction changed across experiments.

Table 1. Results from Elio Conte' ambiguous picture tests

$p(B+)$	$p(A+\|B+)$	$p(B-)$	$p(A+\|B-)$	$p_T(A+)$	$p(A+)$	z	p
0.5556	0.6000	0.4444	0.3750	0.5000	0.6667	2.0287	0.0425
0.6207	0.7776	0.3793	0.5425	0.6884	0.5517	-2.1448	0.0320
0.3529	0.1667	0.6471	0.6364	0.4706	0.6471	2.0722	0.0382

Categorization - decision making

Townsend and colleagues (Townsend, 2000) introduced a new paradigm to study the interactions between categorization and decision making, which we discovered is highly suitable for investigating interference effects. On each trial, participants were shown pictures of faces, which varied along two dimensions (face width and lip thickness). Two different distributions of faces were used: on average a 'narrow' face distribution had a narrow width and thick lips; on average a 'wide' face distribution had a wide width and thin lips. The participants were asked to categorize the faces as belonging to either a 'good' guy or 'bad' guy group, and/or they were asked to decide whether to take a 'attack' or 'withdraw' action. The participants were informed that 'narrow' faces had a .60 probability to come from the 'bad guy' population, and 'wide' faces had a .60 chance to come from the 'good guy' population. The participants were usually rewarded for attacking 'bad guys' and they were usually rewarded for withdrawing from 'good guys.' The primary manipulation was produced by using the following two test conditions, presented across a series of trials, to each participant. In the C-then-D condition, participants made a categorization followed by an action decision; in the D-Alone condition, participants only made an action decision.

The categorization - decision paradigm provides a simple test of the law of total probability. In particular, this paradigm allows one to compare the probability of taking an 'attack' action obtained from the D-Alone condition with the total probability computed from the C-then-D condition. Townsend et al. (2000) reported chi square tests at the .05 significance level. They found that with narrow faces, 38 out of 138 participants produced statistically significant deviations; with wide faces, 34 out of 138 did so. These numbers are much higher than what is expected by chance alone (using a significance level at .05 the expected number is only $(.05)(138)=6.9$).

Townsend et al. did not report the direction of the interference effects, and only reported the chi square magnitudes. To determine the direction of the interference effects, Wang (Busemeyer, Wang, Mogilianki-Lambert, 2009) conducted a replication of the Townsend study using 26 participants, but each participant provided 51 observations for the C-D condition for a total of 26x51=1326 observations, and each person produced 17 observations for the D-alone condition producing $17\times 26=442$ total observations. The results are shown in Table 2.

Table 2. Category-decision making task results

Type	$p(G)$	$p(A\|G)$	$p(B)$	$p(A\|B)$	$p_T(A)$	$p(A)$	t	p
Wide	.84	.35	.16	.52	.37	.39	.5733	.5716
Narrow	.17	.41	.83	.63	.59	.69	2.54	.018

The column labeled $p(G)$ shows the probability of categorizing the face as a 'good guy,' the column labeled $p(A|G)$ shows the probability of attacking given that the face was categorized as a 'good guy,' the column $p(A|B)$ shows the probability of 'attacking' given that the face was categorized as a 'bad guy', the column $p_T(A)$ shows the total probability, and the column $p(A)$ shows the probability of 'attack' when this decision was made alone. The results for both faces produce some deviation between $p_T(A)$ and $p(A)$ but the most pronounced deviation occurred for the narrow faces, which produced a large positive interference effect. The interference effect was statistically significant for the narrow faces, but not for the wide faces (using a paired t-test).

Disjunction effect in decision making

The earliest direct report of interference effects is the disjunction effect (Tversky and Shafir, 1992). The original studies were designed to test a rational axiom of decision theory called the sure thing principle (Savage, 1954). According to the sure thing principle, if under state of the world X you prefer action A over B, and if under the complementary state of the world ~X you also prefer action A over B, then you should prefer action A over B even when you do not know the state of the world. Tversky and Shafir experimentally tested this principle by presenting students with a two stage gamble, which is a gamble which can be played twice. At each stage the decision was whether or not to play a gamble that has an equal chance of winning $200 or losing $100 (the real amount won or lost was actually $2.00 and $1.00 respectively). The key result is based on the decision for the second play, after finishing the first play. The experiment included three conditions: one in which the students were to assume that they already won the first gamble, a second condition in which they were to assume that they lost the first gamble, and a third in which they didn't know the outcome of the first gamble. If they thought they won the first gamble, the majority (69%) chose to play again; if they thought they lost the first gamble, then again the majority (59%) chose to play again; but if they didn't know whether they won or lost, then the majority chose not to play (only 36% wanted to play again). Tversky and Shafir replicated this experiment using both a within subject design (the same person made choices under all conditions separated by a week) as well as with a between subject design (different groups of participants received known and unknown conditions).

This result can be interpreted as an interference effect for the following reason. Define G as the event of playing the second gamble, W is observing a win on the first gamble, and L is observing a loss on the first gamble. The player can choose G or $\bar{G}$ alone (without measuring the outcome of the first game); or the player can *observe* the outcome (W,L) first and then choose G or $\bar{G}$. Then $p(G)$ is the probability of gambling under the unknown condition, and $p_T(G) = p(W)\cdot p(G|W) + p(L)\cdot p(G|L)$ is the probability of choosing to gamble after observing the first play outcome. The total probability is a weighted average of the two known conditions, and so it requires that the probability of playing under the unknown condition must lie in between the two known probabilities. The results show that the probability for the unknown condition is below the smaller probability for the known condition. Therefore we have $p(G) < p(G|L) < p_T(G)$, which implies a negative interference effect.

This result is cited quite frequently, but the result remains controversial. Note that the gamble is actually quite attractive and it has a very positive expected value. Barkan and Busemeyer conducted a very similar study using the same gamble and under conditions in which participants chose to play the gamble a second time under three conditions: planning for a win or planning for a loss or without planning at all for the outcome of the first gamble; but the participants always preferred to play the gamble about 70% of the time and no disjunction effect was found (Barkan and Busemeyer, 1999). Another study attempted a direct replication of Tversky and Shafir's gambling study, but they failed to find a disjunction effect (although they did find a reduction of choice for the gamble under the unknown condition as compared to the known win condition) (Kuhberger, 2001).

Another paradigm that Tversky and Shafir used was the vacation problem. The experiment asked 200 students to imaging trying to decide whether to purchase or wait to purchase a vacation ticket before an exam test result is known. A third of the students were told to imagine that they passed the exam, and 54% these wanted to purchase the ticket; also a third were told to imagine failing the test, and 57% of these wanted to purchase the ticket; but for the last third of the participants that were told nothing about the exam, only 32% wanted to purchase the ticket. This disjunction effect again produces a negative interference effect. However, only one study tried to replicate this and this study concluded that the results were produced by an artifact in the specific way the questions were asked and thus questioned whether the results actually test the sure thing principle (Bagassi and Macchi, 2006).

A third paradigm, the prisoner dilemma paradigm, was used by Shafir and Tversky (1992) to test the sure thing principle. In all PDs, the two players need to decide independently whether to cooperate with the other player or to defect against the other. The player who stands to gain the most is the one who defects against a cooperating player. Mutual cooperation yields the second-highest payoff for each player. Mutual defection gives the players a payoff lower than that gained from mutual cooperation. Finally, the player who cooperates with a defective player gains the least. No matter what the other player does, an individual player always gains more when he defects; this makes defection the dominant option when the game is played only once against a given opponent (one-shot PD). A total of 80 participants were involved and each person played 6 PD games. Shafir and Tversky found that when a player was informed that the opponent defected, then 97% of the time they defected; if the player was informed that the other opponent cooperated, then 84% of the time they defected; but if they didn't know what the opponent chose then only 63% chose to defect. No average of the high percentages from the two known conditions can equal the low average for the unknown condition and so this again is a negative interference effect.

Several other studies were conducted to replicate and extend the disjunction effect using the prisoner dilemma game. The first used 80 participants, each playing 2 PD's, and half were required to predict what the opponent would do and half were not asked to make this prediction. Players who guessed the opponent's action defected a total of 45% of the time; players who were not asked to guess the opponent's choice defected only 22.5% of the time (Croson, 1999). More specifically, participants who guessed that the opponent defected chose to defect 68% of the time; players who guessed that the other player cooperated chose to defect 17% of the time. This is not a violation of the sure thing principle, but it is an interference effect. Croson didn't report the percentage who predicted defection, but we can infer from the previous statistics this to be 53.85%. A later study by Li and Taplan also found evidence for disjunction effects but much weaker than Shafir and Tversky's original study (Li and Taplan, 2002). Most recently, however, a very robust disjunction effect and replication of Shafir and Tversky (1992) was obtained by Matthew (Busemeyer, Matthew, Wang, 2006). A total of 88 students played 6 PD games for real money against a computer agent. When told that the agent defected, then 92.4% defected; when told that the agent cooperated, then 83.6% defected; but when the agent's action was unknown only 64.9% defected. This was a very close replication of Shafir and Tversky's results, and once again no average of the two high percentages from the known conditions can equal the low percentage from the unknown condition. Table 3 provides a summary of all of the prisoner dilemma results.

Table 3. Summary of Disjunction effect with PD

Study	*Known Defect*	*Known Coop*	*Unknown*
Shafir & Tversky	.97	.84	.63
Croson	.68	.17	.225
Li & Taplan	.83	.66	.60
Busemeyer et al.	.91	.84	.66
average	.84	.63	.53

In conclusion, the disjunction effect for the gambling problem and the vacation problem remain in doubt. But the interference effect for the prisoner dilemma game is now well established.

Conjunction and disjunction fallacies

Another line of research that can be interpreted as evidence for interference effects is the conjunction fallacy (Tversky and Kahneman, 1983). For example, judges (e.g. students from Stanford) are provided a brief story about a woman named Linda, who used to be a philosophy student at a liberal university and who used to be active in an anti-nuclear movement. Then the judge is asked to rank the likelihood of the following events: that Linda is now (a) active in the feminist movement, (b) a bank teller, and (c) active in the feminist movement and a bank teller, (d) active in the feminist movement and not a bank teller, (e) active in the feminist movement or a bank teller. The conjunction fallacy occurs when option c is judged to be more likely than option b (even though the latter contains the former). Students also tend to judge the disjunction (e) to be less likely than the individual event (a) which is called the disjunction fallacy (Carlson and Yates, 1989).These effects are very robust effect and they has been found with many different types of stories. The conjunction fallacy is also obtained using betting procedures that do not involve even involve asking directly for probabilities (Sides and Osherson, 2002).

Why is the conjunction error considered an interference effect. To answer this, define F as the event 'yes to feminism,' B as the event 'bank teller,' and S is the Linda story. According to the law of total probability

$$p_T(B|S) =$$
$$p(F \cap B|S) + p(\overline{F} \cap B|S) > p(F \cap B|S),$$

but the judgments produce

$$p(B|S) < p(F \cap B|S) < p_T(B|S)$$

which implies a negative interference effect.

Overextension effects in conceptual combinations

Hampton asked students to judge the strength of category membership for various natural items and found that they often rated the membership for a conjunction of two categories to be greater than one of the individual categories (Hampton, 1988). For example when presented the item 'Cuckoo' they rated its strength (on a zero to one scale) for the category pet to be .575; and they rated its strength for the category bird to be 1.0; but they rated its strength for the category 'pet bird' to be .842. This is analogous to the conjunction fallacy described above. In a second study, Hampton found that students often rate the membership strength of an item to a disjunction of two categories to be smaller than the rating for one of the individual categories (Hampton, 1988). For example, when presented with the item 'ashtray' they rated its strength (on a zero to one scale) for 'home furnishings' to be .7; they rated its strength for furniture to be .30; but they rated its strength for 'home furnishings or furniture' to be .25. This is analogous to the disjunction fallacy described above. These and other effects were later replicated by Hampton.

The overextension effect for the conjunction can be viewed as an interference effect by using the following interpretation. Define $p(A|x)$ as the probability that category A is true given the item x, and $p(A \cap B|x)$ as the probability that category $A \cap B$ is true when given the item x. Then according to the law of total probability we have $p(A \cap B|x) < p_T(A|x)$ but the judgments demonstrate

$$p(A|x) < p(A \cap B|x) < p_T(A|x)$$

which again implies a negative interference effect.

Memory recognition over-distribution effect

The phenomenon of interest is observed in human memory experiments that use a memory recognition paradigm called the conjoint -- recognition paradigm (Brainerd, Reyna, Mojardin 1999). Initially, participants are rehearsed on a set T of memory targets (e.g., each member is a short description of an event). After a delay, a recognition test phase occurs, during which they are presented a series of test probes that consist of trained targets from T, related non-targets from a different set R of distracting events (e.g. each member is a new event that has some meaningful relation to a target event), and unrelated set U of non-target items (e.g. each member is completely unrelated to the targets). During the memory test phase, three different types of recognition instructions are employed: the first is a verbatim instruction (V) that requires one to accept only exact targets from T; the second is a gist instruction (G) that requires one to accept only distractors from the related non targets from R; the third is an instruction to accept verbatim or gist items (VorG), that is it requires one to accept probes from either from T or R. Hereafter V represents the event 'accept as a target from T ', G represents the event 'accept as a non target from R and VorG represents the event 'accept as either a target from T or a non target from R. Note that $T \cap R = \emptyset$, and so logically V and G are supposed to be mutually exclusive events. Also, logically the event VorG should equal the event $V \cup G$, but this remains an empirical question.

Consider memory test trials that employ a test probe belonging to the target set T. If the verbatim question is asked, then probability of accepting the target is formally defined by the conditional probability $p(V|T)$; if the gist question is asked, then the probability of accepting the target is formally defined by the probability $p(G|T)$; finally if the verbatim or gist question is asked, then this is formally defined by the probability $p(VorG|T)$.

Logically, a probe x comes from T or G but not both, implying that $p(VorG|T) = p(V|T) + p(G|T)$. The difference $EOD(T) = p(V|T) + p(G|T) - p(VorG|T)$ is an episodic over distribution effect. A positive EOD effect was obtained from 116 different experimental conditions (Brainerd and Reyna, 2008). All but 10% of the 116 studies produced this effect, and the mean value of the EOD equals .18.

III. What are explanations for these effects?

Interference effects are empirical results that need a scientific explanation. One cannot immediately jump to the conclusion that they are evidence for quantum mechanisms. Thomas Young's original interference results were taken as evidence for a classical wave mechanism. Nor can one jump to the conclusion that interference effects are explained psychologically without quantum theory. Psychologists often like to explain results using intuitive conceptual ideas such as 'interference effects simply result from the first question producing a context that affects the second question.' While this may serve as a conceptual description, this same psychological explanation can be formulated mathematically as either a classic or a quantum model, and so it does not discriminate between these two theoretical competitors. The scientific way to determine whether the data follow quantum or classical probability rules is to derive formal predictions from each theory and then compare the predictions with the data. The model that best predicts the experimental results is taken as the best explanation.

Perceptual interference effect

Let us examine a classical versus a quantum explanation for the data appearing in the first condition of Conte (2009) (the first row of Table 1). First consider a very simple Markov model for this task. This model captures the psychological idea that when judgment B precedes A there is some carry over effect of B on A, which does not occur under the A alone condition. Therefore the BA condition changes the states that are entered as compared to the A alone condition. Suppose from an initial start position the person can transit to one of three states, B+, B-, or B0. From each of these three states the person can transit to A+ or A-. For the BA condition, we assume (a) that the transition probability from the start state to state B+ equals $p(B+) = .5556$, and likewise $p(B-) = .4444$ so that $p(B0) = 0$, and (b) we also introduce the carry over effects such that the transition probability from state B+ to state A+ equals $p(A+|B+) = .60$ and the transition probability from state B- to A+ equals $p(A+|B-) = .375$. This of course reproduces the results for the AB order. For the B alone condition, we assume (c) that the state remains in state B0 so that we have $p(B0) = 1$ (no positive or negative state to carry over) and (d) that the transition from B0 to A+ equals $p(A+|B0) = .50$. This reproduces the results for the B alone condition. So this simple Markov model reproduces the first row of Table 1, and the same principles could be used to reproduce the remaining two rows. But of course this model does it in a scientifically unsatisfactory way. First it fails to provide any degrees of freedom for testing the model, and second it provides no mechanism for relating the probabilities for the judgment B alone condition to the judgments for B after judging A in the AB condition. So this is not an acceptable model without some further theoretical constraints and mechanisms.

Conte and colleagues interpreted their results using a quantum model (Conte et al., 2009). We can describe the basic idea as follows. First consider the BA order and suppose the person starts out a state $|S\rangle$. Then there is an amplitude $\langle B_+|S\rangle$ of transiting to the positive interpretation for B and another amplitude $\langle B_-|S\rangle$ of transiting to the negative interpretation, with $|\langle B_+|S\rangle|^2+|\langle B_-|S\rangle|^2=1$. On the one hand, if the state transits to the positive interpretation for B, then there is an amplitude $\langle A_+|B_+\rangle$ of transiting to the positive interpretation for A, and another amplitude $\langle A_+|B_+\rangle$ of transiting to the negative interpretation of A, $|\langle A_+|B_+\rangle|^2+|\langle A_-|B_+\rangle|^2=1$. On the other hand, if the state transits to the negative interpretation of B, then there is an amplitude $\langle A_+|B_-\rangle$ of transiting to the positive interpretation of A, and another amplitude $\langle A_-|B_-\rangle$ of transiting to the negative interpretation of A, with $|\langle A_+|B_-\rangle|^2+|\langle A_-|B_-\rangle|^2=1$. According to quantum probability theory, the total probability for the positive A interpretation (after first resolving the B interpretation) is computed as

$$p_T(A+)=|\langle A_+|B_+\rangle|^2\cdot|\langle B_+|S\rangle|^2+|\langle A_+|B_-\rangle|^2\cdot|\langle B_-|S\rangle|^2.$$

This part of the quantum model is almost the same as the Markov model described above except that it uses squared amplitudes for probabilities. Why do that? Well the answer is that these same amplitudes are used again to explain the probability of a positive interpretation to A alone (leaving the B interpretation unresolved). According to the quantum model, this equals

$$p(A+)=|\langle A_+|S\rangle|^2=|\langle A_+|B_+\rangle\langle B_+|S\rangle+\langle A_+|B_-\rangle\langle B_-|S\rangle|^2.$$

Let us see how this works for the first condition in Table 1. Using the results from the first condition in Table 1, we set $\langle B_+|S\rangle=\sqrt{.5556}$, $\langle A_+|B_+\rangle=\sqrt{.6}$, and $\langle A_+|B_-\rangle=\sqrt{.375}\cdot e^{i\cdot\theta}$, with $\theta=1.2094$ (a value that is solved to fit the interference effect). Then we perfectly reproduce all of the results for the first condition in Table 1. A similar method with new assignments for the transition amplitudes is used to reproduce all three conditions. Thus the same amplitudes are used to explain both the probabilities for the B alone judgments and the probabilities for the B judgments when they follow judgments for A.

There is a more important advantage to the quantum model. That is it makes a new empirically testable prediction. The transition amplitudes for the quantum model must satisfy a unitary requirement, which implies a double stochasticity constraint on the transition probabilities (Conte et al. 2009): $p(A+|B+)=p(A-|B-)$ and $p(A-|B+)=p(A+|B-)$. This provides a single degree of freedom for testing the model. In fact, this test passes with great accuracy for the example in the first row of Table 1. However, it does not succeed for the next two conditions in Table 1, which indicates that perhaps a more complex representation is needed (such as representation with non orthogonal basis vectors as proposed by Khrennikov, 2010, section 4.4).

Category decision making interference effects

Townsend *et al.,* originally proposed a Markov model for the task: the person starts in a state determined by the face denote F. Then there is a probability $p(G|F)$ of transiting to the 'good guy' category and another probability $p(B|F)=1-p(G|F)$ of transiting to the category 'bad guy.' On the one hand, if the state transits to 'good guy' then there is a probability $p(A|G)$ of attacking, and probability $p(W|G)$ of withdrawing. On the other hand, if the state transits to the 'bad guy' category, then there is a probability $p(A|B)$ of attacking and another probability $p(W|B)$ of withdrawing. This model satisfies the law of total probability $p(A|F) = p(G|F)\cdot p(A|G) + p(B|F)\cdot p(A|B)$ and so it cannot explain the interference effect.

Another classic psychology model is the multidimensional signal detection model used in general recognition theory (Ashby and Townsend, 1986). According to this theory, on each trial, the presentation of a face produces a perceptual image, which is represented as a point within a two multidimensional (face width, lip thickness) perceptual space. Furthermore, each point in the perceptual space is assigned to a 'good' guy (denoted G) or 'bad' guy (denoted B) category response label; and at the same time, each point is also assigned a 'withdraw' (denoted W) or 'attack' (denoted A) action. Let $G\&W$ represent the set of points in the space that are assigned to the 'good' guy category and the 'withdraw' action; and analogous definitions apply to form the sets $G\&A$, $B\&W$, and $B\&A$. Thus the probability of categorizing the face as a 'good guy' and taking a 'withdraw' action, denoted $p(G\&W)$, equals the probability of sampling a face that belongs to the $G\&W$ set of points; the other three probabilities, $p(G\&A)$, $p(B\&W)$, and $p(B\&A)$, are determined in an analogous manner. But once again, the marginal probability of taking a 'defensive' action is determined by the law of total probability:

$$p(A) = p(G\&A) + p(B\&A)$$
$$= p(G)\cdot p(A|G) + p(B)\cdot p(A|B).$$

What is a quantum model for this task? Once again we can use the simple quantum model described earlier for the perceptual ambiguity experiments. The person starts in a state $|F\rangle$ determined by the face. Then there is an amplitude $\langle G|F\rangle$ of transiting to the 'good guy' category and another amplitude $\langle B|F\rangle$ of transiting to the 'bad guy' category, $|\langle G|F\rangle|^2 + |\langle B|F\rangle|^2 = 1$. On the one hand, if the state transits to the 'good guy' then there is an amplitude $\langle A|G\rangle$ of transiting to the 'attack' action and another amplitude $\langle W|G\rangle$ of transiting to the 'withdraw' action, $|\langle A|G\rangle|^2 + |\langle W|G\rangle|^2 = 1$. On the other hand, if the state transits to the 'bad guy' category, then there is an amplitude $\langle A|B\rangle$ of transiting to the 'attack' action and another amplitude $\langle W|B\rangle$ of transiting to the 'withdraw' action, $|\langle A|B\rangle|^2 + |\langle W|B\rangle|^2 = 1.$ According to quantum probability theory, the total probability for attack (after first resolving the category) is computed the same as before $p_T(A|F) = |\langle A|G\rangle|^2\cdot|\langle G|F\rangle|^2 + |\langle A|B\rangle|^2\cdot|\langle B|F\rangle|.$

This looks a lot like the Markov model so far. But the probability to attack (leaving the category unresolved) equals equals $p(A|F) = |\langle A|F\rangle|^2 = |\langle A|G\rangle\langle G|F\rangle + \langle A|B\rangle\langle B|F\rangle|^2$.

This violates the law of total probability and produces interference effects.

Let us see how this works for the narrow face condition in which we observed the largest interference effect. Suppose we set

$\langle G|F\rangle = \sqrt{.17}$, $\langle A|G\rangle = \sqrt{.41}$,

and $\langle A|B\rangle = \sqrt{.63}\cdot e^{i\theta}$, $\theta = 1.3128$,

then we exactly reproduce all the results for the narrow face in Table 2. A similar method can be used to reproduce the wide face data.

What about the test of the double stochastic property implied by the unitary requirement for the simple quantum model? This test is passed with very good accuracy for the narrow face condition, but it fails for the wide face condition. To accommodate the latter result, Busemeyer *et al.*, (2009) needed to formulate a four dimensional quantum model.

We have not proven that the quantum model is right for this finding, or even that the quantum model is better than all Markov models or all signal detection models. For example, we could construct other higher dimensional Markov models to fit these results. We could also allow response boundaries to change across tasks in the signal detection models. What we can say with the data so far is say that two of the most popular traditional models for this task (both based on classic probability theory) fail to explain the results. The simple quantum model can fit the results of the narrow face very well, but it needs to be extended to accommodate the violation of the unitary property for the wide face data.

Disjunction effect

The original explanation for the disjunction effect was a psychological explanation based on the failure of consequential reasoning under the unknown conditions. Shafir and Tversky (1992) explained the finding in terms of choice based on reasons as follows. Consider for example, the two stage gambling problem. If the person knew they won, then they had extra house money with which to play and for this reason they chose to play again; if the person knew they had lost, then they needed to recover their losses and for this other reason they chose to play again; but if they didn't know the outcome of the game, then these two reasons did not emerge into their minds. Why not? If the first play is unknown, it must definitely be either a win or a loss, and it can't be anything else. So the mystery is why these reasons don't emerge for the unknown condition. If choice is based on reasons, then the unknown condition has two good reasons. Somehow these two good reasons cancel out to produce no reasons at all! This sounds a lot like wave interference where one wave is rising and the other is falling.

Kuhberger et al. (2001) formulated a simple two dimensional Markov type of model to explain the Croson results. There is some probability $p(GD)$ to guess the opponent will defect and some probability to guess the opponent will cooperate $p(GC) = 1 - p(GD)$. The player matches the guessed behavior of the opponent so that if the opponent is predicted to defect, then the player defects, that is $p(D|GD) = 1-\varepsilon$ and if the opponent cooperates then the player cooperates too so that $p(C|GC) = 1-\varepsilon$ for small ε. Using Croson's estimate for $p(GD) = .54$ then we obtain $P(D)=.54\cdot(1-\varepsilon)+.46\cdot\varepsilon > .50$ (with small ε) for the total probability of defecting when asked to guess, which is higher than the observed probability equal to $.45$. Clearly it is wrong to to assume $p(D|GD) \approx 1$ and $p(C|GC) \approx 1$ because we can see from Shafir and Tversky's original study (for the opponent known to defect and opponent known to cooperate conditions) that the probabilities do not obey this rule, $p(D|KD) = .97 \approx 1$ but $p(C|KC) = .16 < 1$. In fact, if we apply Kuhberger et al.'s explanation to the unknown condition in Shafir and Tversky's experiment then the probability of defecting equals the weighted average

$$p(GD)\cdot p(D|GD) + p(GC)\cdot p(D|GC)$$
$$= p(GD)\cdot(.97) + p(GC)\cdot(.84) >$$
$$.63$$

and so this explanation does not seem to work.

Another way to explain the PD results is to assume that the person can enter one of three states: guess the opponent will defect and under this state the player defects with probability $p(D|GD)$; guess the opponent will cooperate, and under this state the player defects with probability $p(D|GC)$; and a confused state $?$ in which $p(D|?) = .50$. We can assume that when the opponent action is known, then the probability of entering the confused state is small, say zero for example; but when the opponent's action is unknown the probability is high, say it is θ. Then the probability of defecting when the opponent is known to defect equals $p(D|KD)$, and the probability of defecting when the opponent is known to cooperate is $p(D|KC)$, but the probability of defecting in the unknown case equals

$$p(D|UK) = (1-\theta)\cdot[p(GD)\cdot p(D|GD) + p(GC)\cdot p(D|GC)] + (.5)\cdot\theta$$

For large θ this can explain the disjunction effect for the PD game. Croson tried to rule out this explanation in one of her experiments, but there is an even stronger reason for rejecting this idea. The confused state model must predict that the unknown condition causes the choice probabilities to converge from either direction toward .50; however in the original Tversky and Shafir (1992) disjunction effect, the unknown state produced a probability that started above .50 and then crossed well below .50 down toward zero. This result cannot be explained by the confused state model, but this result has not been replicated either.

This psychological explanation given by Shafir and Tversky (1992) is quite consistent with a formal quantum mechanism for the effect. Busemeyer *et al.* (2006) originally suggested a quantum interference interpretation for the disjunction effect, and since that time, various quantum models for this effect have been proposed, each one ultimately explaining the effects by interference terms, which includes (Accardi, Khrennikov, Ohya, 2009), (Pothos and Busemeyer, 2009), (Khrennikov and Haven, 2009), Aerts (2009), and Yukalov and Sornette (2010).

Let us try once again with a simple quantum model devised by Khrennikov and Haven (2009) for this result. The person starts in a state $|S\rangle$. Then there is an amplitude $\langle O_C|S\rangle$ of transiting to the 'opponent cooperates' inference and another amplitude $\langle O_D|S\rangle$ of transiting to the 'opponent defects' inference, $|\langle O_C|S\rangle|^2+|\langle O_D|S\rangle|^2=1$. On the one hand, if the state transits to the 'opponent cooperates' inference then there is an amplitude $\langle C|O_C\rangle$ of transiting to the 'cooperate' action and another amplitude $\langle D|O_C\rangle$ of transiting to the 'defect' action, $|\langle C|O_C\rangle|^2+|\langle D|O_C\rangle|^2=1.$ On the other hand, if the state transits to the 'opponent defects' inference, then there is an amplitude $\langle C|O_D\rangle$ of transiting to the 'cooperate' action and another amplitude $\langle D|O_D\rangle$ of transiting to the defect action, $|\langle C|O_D\rangle|^2+|\langle D|O_D\rangle|^2=1.$ According to quantum probability theory, the total probability for defecting (after first resolving the inference about the other player) is computed as $p_T(D)=|\langle D|O_D\rangle|^2\cdot|\langle O_D|S\rangle|^2+|\langle D|O_C\rangle|^2\cdot|\langle O_C|S\rangle|.$ But the probability to defect (leaving the inference about the other player unresolved) equals $p(D)=|\langle D|S\rangle|^2=|\langle D|O_D\rangle\langle O_D|S\rangle+\langle D|O_C\rangle\langle O_C|S\rangle|^2$.

Let us see how this works for the average PD data in Table 3. Suppose we set $\langle O_C|S\rangle=\sqrt{.50}$ (this is a reasonable estimate from Croson's experiments), $\langle D|O_C\rangle=\sqrt{.63}$, and $\langle D|O_D\rangle=\sqrt{.84}\cdot e^{i\theta}$, $\theta=1.8565$, then we exactly reproduce the results for the average PD data in Table 3.

What about the test of the unitary property for this data. It fails dramatically in this case. For this reason, Khrennikov (2009) employed a 2 dimensional model but using non orthogonal basis states.

Conjunction fallacy

The conjunction effect was first given a psychological explanation, called the representativeness heuristic, by Tversky and Kahneman (1983). The idea is that one matches the features of the story with a prototype representing the category and picks the best match. The match between the story and the prototype for feminist bank teller is greater than the match between the story and the prototype for bank teller, which presumably explains the effect. However, conjunction errors are obtained with unrelated items (Gavanski and Roskos-Ewoldsen, 1991). For example, suppose one story is about a boring engineer named Bill, and another story is about a liberal philosophy student named Linda. Then people tend to judge 'Bill is a jazz player and Linda is a feminist' to be more likely than 'Bill is a jazz player.' This presumably argues against the representativeness heuristic because there is no single prototype that one can form for the 'Bill is a jazz player and Linda is a feminist' question.

A second explanation for the conjunction fallacy is that people reverse the conditional so that instead of judging $p(F\cap B|S)$ they judge $p(S|F\cap B)$. There is no violation of classic probability theory for the latter interpretation and Linda is a likely story given the Feminist bank teller category. However, this model fails to explain why the disjunction is always judged higher than the conjunction because there is no way to order $p(S|F\cup B)$ versus $p(S|F\cap B)$.

A third popular explanation for conjunction errors in psychology is that people estimate the conjunction by averaging (rather than multiplying) the probabilities: $J(A\ and\ B)=w\cdot p(A)+(1-w)\cdot p(B), \quad 0\le w\le 1.$ The average of a high and low likelihood will always be in between these two extremes and thus the average produced by the conjunction of two events exceeds the likelihood of the lower event. Disjunction errors are explained using the same averaging model except that different weights are applied for the two tasks. More weight is given to the lower probability event for conjunctions, and more weight is given to the higher probability event for disjunction.

Franco (2007) originally proposed the idea of using the product of non-commuting quantum operators to represent the sequence of questions involved in a conjunction question. On this basis he derived a quantum model using cross product interference terms to explain the conjunction fallacy (Franco, 2009). Subsequently several variations of quantum models using interference to explain the conjunction fallacy have been proposed including Khrennikov (2010), Conte (2009), and Yukalov and Sornette (2009). Recently we have developed a general model for both the conjunction and disjunction fallacies and a number of other probability judgment errors (Busemeyer *et al.,* 2009) building on Franco's original ideas.

To quickly see how the quantum model applies to this problem, consider once again our simple model used earlier. Let F represent a positive answer to the feminist question and $\bar{F}$ represents a negative answer, and B represents a positive answer to the bank teller question. The Linda story creates a state vector $|S\rangle$. Starting from this state, the judge can transit to the positive answer to the feminist question with an amplitude $\langle F|S\rangle$. After transitioning to the positive feminist state, the judge can transit to the positive bank teller state with amplitude $\langle B|F\rangle$. Therefore the probability of saying yes to the conjunction equals $|\langle B|F\rangle\langle F|S\rangle|^2$. Now consider the probability to the bank teller question alone. This is determined by the direct transition from the story to bank teller:

$$|\langle B|S\rangle|^2 = |\langle B|F\rangle\langle F|S\rangle + \langle B|\bar{F}\rangle\langle\bar{F}|S\rangle|^2 \ .$$

For example, suppose we set

$$\langle F|S\rangle = \sqrt{.85}, \quad \langle B|F\rangle = \sqrt{.25},$$

$$\text{and } \langle B|\bar{F}\rangle = \sqrt{.3}\cdot e^{i\theta}, \quad \theta = 2.507.$$

Then we obtain

$$|\langle B|F\rangle\langle F|S\rangle|^2 = .2125 > |\langle B|S\rangle|^2 = .10,$$

which reproduces the conjunction fallacy. Psychologically, this explanation says that it is almost impossible to transit directly from the state created by the story to a positive answer to the bank teller question. But if the person first considers whether Linda is a feminist (which is likely to be yes) and then thinks about possible occupations for feminists (one could be bank teller) then it become possible to reach the conclusion that Linda is a feminist bank teller. The disjunction fallacy is explained using the same principles because the probability of saying yes to the disjunction equals the probability of saying no to the conjunction 'Linda is not a bank teller and Linda is not a feminist.' Thus the same exact model is used for both types of fallacies.

One serious problem for the averaging model is that it cannot explain dependence between events. The effect of event A in the combination of A with B is predicted to be exactly the same as the effect of A with the combination of A with B.' However, if there are strong dependencies between these events, then the effect of A is observed to be quite different depending on whether it is paired with B or B' event (Miyamoto, 1995). The quantum models are influenced by these dependencies and have an advantage over the averaging model with respect to this issue.

Disjunction errors in conceptual combinations

The averaging model is a very popular model in psychology, and it is also used to explain over-extension effects found with conceptual combinations. Aerts and Gabora have conducted a long program of research developing quantum models to explain the conceptual combination data. At the beginning of this research program, it was assumed that the category (concept) was represented as state vector in a Hilbert space, and the item was a context (measurement) operator defined in the same space (Aerts and Gabora, 2005). This produced the effect of computing the probability of the item given the category, and this produces $p(x|A\cap B)$ which can exceed $p(x|A)$. Later work reversed the roles so that the item corresponded to a state vector in the Hilbert space and the category was represented by a measurement operator defined in the same space (Aerts, 2009). In the latter model, a combination of categories A, B (conjunction or disjunction of these two categories) is a new emergent concept produced by a superposition of the individual concepts, and the measurement operator for the combination concept equals the outerproduct formed by this superposition state. Then interference terms produced by the superposition are used to explain the overextension effects found with conjunctive concepts.

One major difficulty for an averaging model explanation for overextension effects in conceptual combinations is that they are not always found, and many concepts obey the rules of classic probability theory. To explain this mixture of classic concepts as well as concepts that produce overextension effects, Aerts (2009) proposed a Fock space that is the direct sum of (1) a tensor product space that produces classical probability results, and (2) a space that includes the emergent concepts which account for the overextension effects. However, there is an alternative explanation which does not use quantum theory. One could assume a mixture model in which some proportion of people use an averaging model to form conceptual combinations, and the remaining proportion of people use a classic probability model to judge conceptual combinations. This could mimic the Fock space model and produce both classical and overextension behavior.

We already worked out a simple example of a quantum model for conjunction questions when discussing the conjunction fallacy. Now let us work out a simple example for the disjunction question in the context of a conceptual membership task. Suppose a person is asked if item x is a member of the disjunction A or B and suppose B is more likely than A. This is the same as saying no to the conjunction ‘ x is not in A and x is not in B .’ The person starts in a state $|S_x\rangle$, and the probability of ‘item x not in A and not in B' equals $|\langle\bar{B}|\bar{A}\rangle\langle\bar{A}|S_x\rangle|^2$. The probability of the disjunction is then equal to $1-|\langle\bar{B}|\bar{A}\rangle\langle\bar{A}|S_x\rangle|^2$. The probability of saying that x is a member of B equals $|\langle B|S_x\rangle|^2=1-|\langle\bar{B}|S_x\rangle|^2$. A disjunction violation occurs when $p(A\cup B|x)<p(B|x)$ and for our quantum model this corresponds to

$$1-|\langle\bar{B}|\bar{A}\rangle\langle\bar{A}|S_x\rangle|^2<1-|\langle\bar{B}|S_x\rangle|^2$$

or what is equivalent

$$|\langle\bar{B}|\bar{A}\rangle\langle\bar{A}|S_x\rangle|^2>|\langle\bar{B}|S_x\rangle|^2$$
$$=|\langle\bar{B}|\bar{A}\rangle\langle\bar{A}|S_x\rangle+\langle\bar{B}|A\rangle\langle A|S_x\rangle|^2.$$

So for example, if we set $\langle\bar{A}|S_x\rangle=\sqrt{.85}$, $\langle\bar{B}|\bar{A}\rangle=\sqrt{.25}$, and $\langle\bar{B}|A\rangle=\sqrt{.3}\cdot e^{i\theta}$, $\theta=2.507$. Then we obtain

$$|\langle\bar{B}|\bar{A}\rangle\langle\bar{A}|S_x\rangle|^2=.2125>|\langle\bar{B}|S_x\rangle|^2=.10,$$

which implies the probability that item x belongs to the disjunction of A or B equals $1-|\langle\bar{B}|\bar{A}\rangle\langle\bar{A}|S_x\rangle|^2=1-.2125=.7875$ and this is less than $1-.10=.90=|\langle B|S_x\rangle|^2$ which is the probability that the item x belongs to category B .

Over-distribution in memory recognition

To explain the over distribution effect, (Jacoby, 1991) proposed a dual process model of memory recognition which was later extended by Brainerd and Reyna (2008), and the latter is presented here. This is a Markov model that posits three states. Let us focus on test probes that are true targets from T . In this case, the person may successfully recollect the target (denoted by state S) with probability p_S, or fail to recollect but instead enter a familiar state F with probability p_F, or the person may not recollect and not be familiar. According to this model, the probabilities of responses to questions for a probe that is a true target item from T are given by

$$p(V|T)=p_S+(1-p_S)\cdot p_F$$
$$p(G|T)=(1-p_S)\cdot p_F$$
$$p(VorG|T)=p_S+(1-p_S)\cdot p_F$$

The first equation assumes that either the person will recollect the true target from T or not recollect it but consider it sufficiently familiar to accept (although the latter is inconsistent with the instruction). The second equation assumes the person does not recollect the item but considers it sufficiently similar to accept it as a non target but related item. The third equation is interpreted in the same way as the first.

Obviously, the dual process model predicts episodic over extension because $EOD(T) = p(V|T) + p(G|T) - p(VorG|T) = (1-p_S)\cdot p_F$. It can also produce sums of $p(V|T) + p(G|T)$ that exceed unity (e.g., set $p_S = .10$ and $p_F = .90$). If we interpret $p(VorG|T)$ as the disjunctive probability $p(V \cup G|T)$ of categorizing the target as verbatim or gist, then the dual process model violates classic probability theory. This is because according to the dual process model $p(VorG|T) = p(V|T)$, but classic probability theory requires $P(V \cup G|x) \geq P(V|x)$ and in this case the inequality is strict. However, the dual process model incorrectly predicts that $p(VorG|T) = p(V|T)$ when in fact it is found that $p(VorG|T) > p(V|T)$. Bias terms were introduced into the model by Brainerd and Reyna (2008) to fix this problem.

Recently Busemeyer and Trueblood (2010) developed a quantum model for this phenomena. To be more accurate, they simply applied the quantum mode previously developed for the disjunction fallacy found in probability judgment errors to this new problem. Thus the same model provides an explanation for these two different experimental paradigms. Let us consider our simple quantum model example once again for the $VorG$ question. Given an item from set T, the person starts in a state $|S_T\rangle$. According to the simple quantum model, $p(G|T) = |\langle G|S_T\rangle|^2$, and

$$p(V|T) =$$
$$1 - |\langle \bar{V}|S_T\rangle|^2 = 1 - |\langle \bar{V}|G\rangle\langle G|S_T\rangle + \langle \bar{V}|\bar{G}\rangle\langle \bar{G}|S_T\rangle|^2,$$

and we previously worked out the theoretical prediction for $p(VorG) = 1 - |\langle \bar{V}|\bar{G}\rangle\langle \bar{G}|S_T\rangle|^2$. Suppose we set

$$\langle G|S_T\rangle = \sqrt{.15}, \quad \langle \bar{V}|\bar{G}\rangle = \sqrt{.25},$$

and $\langle \bar{V}|G\rangle = \sqrt{.3}\cdot e^{i\theta}, \quad \theta = 2.507.$

Then we obtain

$$p(G|T) = .15,\ p(VorG|T) = .7875,\ p(V|T) = .90$$

which produces

$$EOD(T) = p(V|T) + p(G|T)$$
$$-p(VorG|T) = .15 + .90 - .7875 = .2625.$$

An advantage of the quantum model is that it naturally accounts for the fact that $p(VorG|T) > p(V|T)$ whereas the dual process model must achieve this by adding bias parameters.

IV. What next?

Quantum explanations for interference effects found in psychology have already made one really important contribution. Quantum theory has provided a common way to understand a number of paradoxical findings that have never been connected before, nor even mentioned together in the same articles. Previous explanations for conjunction and disjunction fallacies have never mentioned the disjunction effect found in the prisoner dilemma game. Memory recognition researchers have never tried to connect their results on over distribution effects with findings from research on conceptual combinations. None of the above research lines have tried to relate their results to interference effects in perception or in categorization. In fact, little if any cross referencing occurs between articles on the conjunction and disjunction fallacies and articles on conceptual combination! By examining interference effects and providing a common quantum account of these effects, quantum theorists have organized a new and general and uniform way to think about all these seemingly unrelated problems. This is a big step forward.

We think that these initial promising steps made toward understanding all of the various interference effects are encouraging other researchers to begin examining quantum models in other applications in psychology, and to design new experimental tests of the models in the areas already reviewed above. These new experimental tests are needed to establish the explanatory rather than descriptive power of quantum models (Blutner, 2010). Enough evidence has been obtained to indicate that quantum theory provides a viable new theoretical approach to modeling research in cognition and decision making. So the answer we give to the question raised by Conte in his lead article *'On the possibility that we think in a quantum probabilistic manner'* is that this is an exciting new approach holding great promise for future research.

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